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VOL. 28, No. 1, Sept.-Oct., 1954



# MATHEMATICS magazine

# MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S. T. Sanders.





*Address* editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to a member of the executive committee.

Manuscripts should be typed on 8½" x 11" paper, double-spaced with 1" margins. We prefer that, in technical papers, the usual introduction be preceded by a *foreword* which states in simple terms what the paper is about.

The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are 1 yr. $3.00; 2 yrs. $5.75; 3 yrs. $8.50; 4 yrs. $11.00; 5 yrs. $13.00. Sponsoring subscriptions are $10.00; single copies 65¢. Reprints, bound, ½¢ per page plus 10¢ each, (thus 25 ten page reprints would cost $1.25 plus $2.50 or $3.75) provided your order is placed before your article goes to press.

Subscriptions and related correspondence should be sent to Inez James, 14068 Van Nuys Blvd., Pacoima, California.

Advertisers should contact The Lees Agencies, P.O. Box 906, Hollywood 28, California.

Entered as second-class matter March 23, 1948 at the Post Office, Pacoima, California under act of congress of March 8, 1876.

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## Trickies

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## COMMENT ON BIRNBAUM AND OMMIDVAR'S "THE GROUP METHOD"*

H. W. Becker

This method is standard equipment in electronics lab courses, as taught by the writer at Mare I. Navy Yard and Omaha, and prob'ly the world over. The optimum group size is 3, the actual number depending on the ratio of manpower to gear. It is necessary that the groups be of the same average ability and speed, so all keep pace with the daily schedule of theory briefings on the experiments. The men group themselves, usually a cluster of "greenhorns" around each "sharpy." But sometimes a couple of experienced "hams" will team up, and tend to outstrip the field; it suffices to suggest they break it up and choose some other partner, to stay in phase with the class.

Strangely enough, it never occurred to the writer to adapt this method to math. instruction. Birnbaum and Ommidvar deserve medals for their innovation and their write-up of its merits could hardly be improved.

* Mathematics Magazine, 28 (1955) 277-9.

## CALCULATING MACHINE - ABACUS

A. M. Maish

Managing Editor
Mathematics Magazine

Dear Sir:

In the November-December 1954 issue, (Volume 28, No. 2) on page 83, there is an article by Cpl. Jerry Adler that contains some errors of fact. I was present at the contest in November 1946 in the Ernie Pyle theater. In fact, I supplied the calculating machine, a model S-10 Friden, that had been through the war and had been reconditioned. I did not, however supply the operator. Due to the rapid turnover of personnel I had nobody available with as much as one month's practice on the machine, so I yielded to another organization.

The Finance Office supplied a corporal who was careful and slow. He had been taught to check and double check. During the contest, he would read one digit, turn his head to the machine, press the correct key, and so on. Sometimes he was still confirming his input when "Hands" Matzuzaki was scribbling the answer with one hand and shaking his abacus with the other to clear the rods for the next problem.

I think anyone in the audience, children included, could have bettered the corporal's performance if given one hour's instruction. The difference in the final result was largely due to difference in ability

# ERRATA IN VOL.28, NO.1

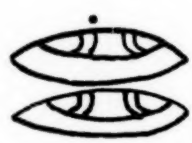
# SEPT.-OCT. ISSUE, 1954

Page 44 - The following figure should stand above the zero following the phrase 'are as follows'

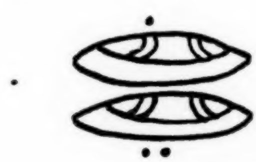
Page 45 - Figures on page 45 are incomplete and should read as follows:

In the space after the phrase 'the second order written'

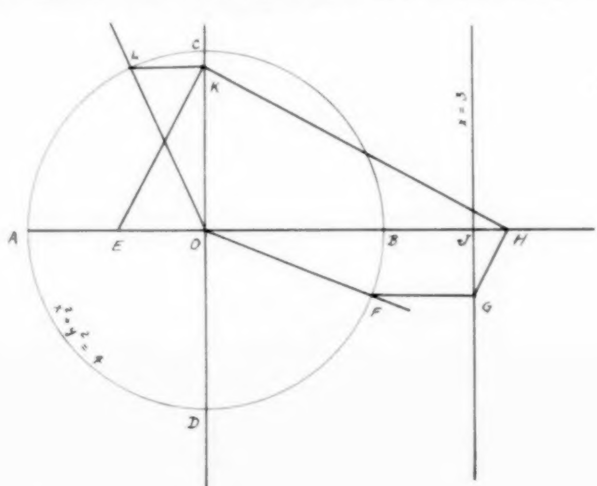
In the space after the phrase 'and is written'



In the space after the phrase 'the number 7,202 is written'



Page 32 - Insert the following figure in the vacant space:



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## OUR CONTRIBUTORS

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*Richard Bellman,* mathematician, the Rand Corporation, Santa Monica, California, was born in New York City in 1920. A graduate of Brooklyn College (B.A. '41) and of Princeton University (Ph.D. '46), he was an assistant professor at Princeton from 1946 - 48 and an associate professor at Stanford University from 1948 - 52. During the war he taught pre-radar electronics for the U.S.A.F., was a mathematician in the Sonar program at Point Loma, California and later was a member of the Special Engineering Division, U.S. Army at Los Alamos. His principal fields of research are differential equations, number theory and dynamic programming.

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Biographies of the other contributors in this issue will appear in the next issue.

# SYSTEMS OF PARTICLES WITH A COMMON CENTROID

Howard Eves

## *INTRODUCTION*

In general, if the members of a system of $n$ weighted particles be independently displaced to new positions, the centroid of the new arrangement will not coincide with the centroid of the original arrangement. It is the purpose of this paper to develop some theorems and constructions concerning rearrangements of $n$ weighted particles for which the centroid remains invariant. Most of the results of the paper will appear as consequences of a few related basic theorems. Generalizations of some notable results of the past will be obtained.

## *THE BASIC THEOREMS*

A well known fundamental theorem on centroids (*cf.* Jeans, *Theoretical Mechanics*, Ginn and Co., 1906, p. 22) may be stated as follows:

THEOREM 1. *If vectors of magnitudes $m_1OB_1, \cdots, m_nOB_n$ act along lines $OB_1, \cdots, OB_n$, then their resultant is of magnitude $(m_1 + \cdots + m_n)OM$ and acts along $OM$, where $M$ is the centroid of particles of masses $m_1, \cdots, m_n$ located at $B_1, \cdots, B_n$.*

From this theorem we shall now obtain the following criterion:

THEOREM 2. *A necessary and sufficient condition for the centroid of a system of particles $A_1, \ldots, A_n$ having masses $m_1, \cdots, m_n$, respectively, to remain invariant when the particles are displaced in space to $A'_1, \cdots, A'_n$ is that the vectors $m_1\overrightarrow{A_1A'_1}, \cdots, m_n\overrightarrow{A_nA'_n}$ form, by parallel displacement, a closed space polygon.*

Let $m$ be the centroid of the $A$-particles. Then, by the fundamental theorem,

$$m_1\overrightarrow{MA}_1 + \cdots + m_n\overrightarrow{MA}_n = 0.$$

To establish the necessity of the condition suppose the vectors $m_1\overrightarrow{A_1A'_1}, \cdots, m_n\overrightarrow{A_nA'_n}$ form, by parallel displacement, a closed space polygon. Then

$$m_1\overrightarrow{A_1A'_1} + \cdots + m_n\overrightarrow{A_nA'_n} = 0.$$

Therefore

$$m_1\overrightarrow{MA'_1} + \cdots + m_n\overrightarrow{MA'_n} = m_1(\overrightarrow{MA}_1 + \overrightarrow{A_1A'_1}) + \cdots + m_n(\overrightarrow{MA}_n + \overrightarrow{A_nA'_n}) = 0,$$

and $M$ is the centroid of the $A'$-particles.

To establish the sufficiency of the condition, suppose

$$m_1 MA_1' + \cdots + m_n MA_n' = 0.$$

Then

$$m_1\overrightarrow{A_1A_1'} + \cdots + m_n\overrightarrow{A_nA_n'} = m_1(\overrightarrow{MA_1'} - \overrightarrow{MA_1}) + \cdots + m_n(\overrightarrow{MA_n'} - \overrightarrow{MA_n})$$

$$(m_1\overrightarrow{MA_1'} + \cdots + m_n\overrightarrow{MA_n'}) - (m_1\overrightarrow{MA_1} + \cdots + m_n\overrightarrow{MA_n}) = 0,$$

and the vectors $m_1\overrightarrow{A_1A_1'}, \cdots, m_n\overrightarrow{A_nA_n'}$ form, by parallel displacement, a closed space polygon.

As an immediate consequence of theorem 2 we have

**THEOREM 3.** *For any space n-gon $B_1B_2\cdots B_n$, we may, by taking $A_1A_1'$ equal and parallel to $B_1B_2/m_1, \cdots, A_nA_n'$ equal and parallel to $B_nB_1/m_n$, displace the system of weighted A-particles into a new system having the same centroid. Since there are $(n-1)!$ space n-gons determined by n points in space, we generally obtain in this way $n(n-1)! = n!$ systems all having the same centroid.*

Another easy consequence of theorem 2 is

**THEOREM 4.** *Let n weighted particles $A_1, \cdots, A_n$ with centroid M be independently displaced in space to the positions $A_1', \cdots, A_n'$. Through M draw $MB_1, \cdots, MB_n$ respectively equal and parallel to $A_1A_1', \cdots, A_nA_n'$. Consider particles located at $B_1, \cdots, B_n$ such that the mass of each B-particle is equal to the mass of the corresponding A-particle. Then the centroid M' of the A'-particles coincides with the centroid of the B-particles.*

We have $\overrightarrow{B_1A_1'} = \overrightarrow{MA_1}, \cdots, \overrightarrow{B_nA_n'} = \overrightarrow{MA_n}$. Therefore

$$m_1\overrightarrow{B_1A_1'} + \cdots + m_n\overrightarrow{B_nA_n'} = m_1\overrightarrow{MA_1} + \cdots + m_n\overrightarrow{MA_n} = 0,$$

by theorem 1. It now follows, from theorem 2, that the centroid of the $B$-particles coincides with the centroid of the $A'$-particles.

## *GENERALIZATION OF McCAY'S THEOREM*

In Book VIII of his *Mathematical Collection*, the great Greek geometer Pappus established the theorem: *If A', B', C' are points on the sides BC, CA, AB of triangle ABC such that $BA'/A'C = CB'/B'A = AC'/C'B$, then triangles A'B'C' and ABC have a common centroid.* Over fourteen hundred years later the French geometer H. Brocard, who is regarded as one of the nineteenth century founders of the modern geometry of the triangle, generalized Pappus' theorem to the following: *If three directly similar triangles BCA', CAB', ABC' be similarly described on the sides BC, CA, AB of triangle ABC, then triangles ABC and A'B'C' have a common*

*centroid.* A surprising number of different proofs have appeared for this theorem of Brocard, and not long after Brocard's accouncement of his theorem it was generalized from a triangle to an arbitrary plane polygon by the Irish geometer W. S. McCay. Some of the proofs of McCay's theorem found in treatments of modern geometry are quite involved. We shall now establish, as a consequence of our theorem 3, a generalization of McCay's theorem.

THEOREM 5. *Let $A_1, \cdots, A_n$ be n weighted coplanar particles of masses $m_1, \cdots, m_n$ respectively. Through $A_1, A_2, \cdots, A_n$ draw, in the plane of the particles, lines $A_1L_1, A_2L_2, \cdots, A_nL_n$ such that $\angle L_1A_1A_2 = \angle L_2A_2A_3 = \cdots = \angle L_nA_nA_1$. Now suppose the particles $A_1, \cdots, A_n$ start to move simultaneously along $A_1L_1, \cdots, A_nL_n$, at rates proportional to $A_1A_2/m_1, \cdots, A_nA_1/m_n$. Then under this motion the centroid of the n particles remains invariant.*

For let $A_1', \cdots, A_n'$ be any set of subsequent positions of the weighted particles. Since $A_1A_1', \cdots, A_nA_n'$ are equally inclined to the corresponding sides $A_1A_2, \cdots, A_nA_1$ of the polygon $A_1 \cdots A_n$, and further, since

$$m_1A_1A_1' : m_2A_2A_2' : \cdots : m_nA_nA_n' = A_1A_2 : A_2A_3 : \cdots : A_nA_1,$$

we may regard $A_1', \cdots, A_n'$ as having been obtained from $A_1, \cdots, A_n$ by displacements equal and parallel to the sides of a polygon directly similar to $A_1 \cdots A_n$. The theorem now follows from theorem 3.

McCay's theorem is the case where $m_1 = \cdots = m_n$.

## *GENERALIZATION OF D'OCAGNE'S THEOREM*

In the Belgian journal *Mathesis*, 1887, p. 265, M. d'Ocagne established the theorem: *If $A', B', C'$ are variable points on the sides AB, BC, CA of triangle ABC, such that $AA' = BB' = CC'$, then the locus of the centroid of the variable triangle $A'B'C'$ is a straight line through the centroid of triangle ABC.* Using our theorem 4 we can easily generalize this theorem of d'Ocagne to the situation of any $n$ arbitrarily weighted particles in space.

THEOREM 6. *Let n weighted particles $A_1, A_2, \cdots, A_n$ in space, with centroid M, be displaced equal distances along $A_1A_2, A_2A_3, \cdots, A_nA_1$ to positions $A_1', A_2', \cdots, A_n'$. Then the locus of the centroid $M'$ of the $A'$-particles is a straight line passing through M.*

Draw $MB_1$ equal and parallel to $A_1A_1', \cdots, MB_n$ equal and parallel to $A_nA_n'$. Then, by theorem 4, the centroid $M'$ of the $A'$-particles coincides with that of a system of similarly weighted $B$-particles. The $B$-particles are cospherical on a sphere of center $M$ and radius $A_1A_1'$. As the distance $A_1A_1'$ changes, the configuration of $B$-particles transforms homothetically with $M$ as center of homothety. It follows that the locus of $M'$ is a straight line through $M$.

D'Ocagne's theorem is the very special case where $n=3$ and all the particles are equally weighted.

### WEILL'S THEOREM AND ITS GENERALIZATION

There is a remarkable theorem due to J. V. Poncelet which may be stated as follows: *If a variable n-gon is inscribed in a conic of a pencil of conics, and if $n-1$ of its sides touch $n-1$ fixed conics of the pencil, then the nth side also touches a fixed conic of the pencil.* If the $n-1$ fixed conics which the $n-1$ sides touch happen to coincide, then it can be shown that the $n$th fixed conic touched by the $n$th side also coincides with them. The result is the well known Poncelet porism: *If two conics are so related that an n-gon can be inscribed in one and circumscribed about the other, then infinitely many n-gons can be so drawn.* The special situation where the pencil of conics is taken as a family of coaxal circles has received much attention. Perhaps the fullest discussion of this special situation is that of A. Weill in *Liouville's Journal*, series 3, vol. IV, 1878. Here Weill established, among other things, the following elegant theorem: *If a variable n-gon is inscribed in one circle and circumscribed about another, then the centroid of a system of equally weighted particles placed at the points of contact of the n-gon's sides with the latter circle remains invariant.* Many ingenious proofs have been supplied for this theorem, especially for the cases $n=3$ and $n=4$. We shall use our theorem 2 to obtain as simple a proof for the general case as any yet given. In fact, we shall use theorem 2 to obtain a simple proof of the following considerable generalization of Weill's theorem.

THEOREM 7. *Consider a variable n-gon $A_1 \cdots A_n$ inscribed in a circle $(O)$ and having its sides $A_1A_2, \cdots, A_nA_1$ touching circles $(O_1), \cdots, (O_n)$ at points $T_1, \cdots, T_n$, where the circles $(O), (O_1), \cdots, (O_n)$ all belong to a common coaxal system. Suppose weighted particles are placed at $T_1, \cdots, T_n$ with masses proportional to $\sqrt{d_1}/r_1, \cdots, \sqrt{d_n}/r_n$, where $r_1, \cdots, r_n$ are the radii of the circles $(O_1), \cdots, (O_n)$, and $d_1, \cdots, d_n$ are the distances from the center of circle $(O)$ to the centers of circles $(O_1), \cdots, (O_n)$. Then the centroid of these weighted particles remains invariant.*

Let $A_1' \cdots A_n'$ be any other position of the variable $n$-gon, and let $T_1', \cdots, T_n'$ be the points of contact of the sides $A_1'A_2', \cdots, A_n'A_1'$ with circles $(O_1), \cdots, (O_n)$. Denote angle $(A_1'A_2', A_1A_2)$ by $\theta_1$ and angle $(A_2A_2', A_1A_1')$ by $\phi_1$.

Now $A_1A_1', \cdots, A_nA_n'$ are all tangent to a circle $(\Omega)$ belonging to the coaxal system, and the points of contact $C_1, \cdots, C_n$ are such that $C_1, T_1, T_1', C_2$ are collinear, $\cdots, C_n, T_n, T_n', C_1$ are collinear (see, *e.g.*, Forder, *Higher Course Geometry*, Cambridge University Press (1949), Cor. 2, p. 224). Denote the radius of $(\Omega)$ by $\rho$ and the distance of the center of

$(\Omega)$ from the center of $(O)$ by $\lambda$. Then

$$T_1T_1'/C_1C_2 = [r_1 \sin(\theta_1/2)] / [\rho \sin(\phi_1/2)].$$

But, applying the law of sines to triangle $T_1A_2C_2$, we see that

$$\sin(\theta_1/2)/\sin(\phi_1/2) = A_2C_2/T_1A_2.$$

Further (see, e.g., Forder, *loc. cit.*, Cor., p. 174), we have

$$A_2C_2/T_1A_2 = \sqrt{\lambda}/\sqrt{d_1}.$$

Therefore

$$T_1T_1'/C_1C_2 = r_1\sqrt{\lambda}/\rho\sqrt{d_1},$$

whence

$$(\sqrt{d_1}/r_1)T_1T_1' = (\sqrt{\lambda}/\rho)C_1C_2.$$

Consequently

$$(\sqrt{d_1}/r_1)T_1T_1' + \cdots + (\sqrt{d_n}/r_n)T_nT_n' = (\sqrt{\lambda}/\rho)(C_1C_2 + \cdots + C_nC_1) = 0,$$

and the theorem now follows by virtue of theorem 2.

The case of Weill's theorem is that where the circles $(O_1), \cdots, (O_n)$ all coincide. Here $d_1 = \cdots = d_n$ and $r_1 = \cdots = r_n$, and the particles have equal masses.

If we take the situation of Weill's theorem and invert with respect to the inscribed circle as circle of inversion we obtain

THEOREM 8. *Consider a variable polygon inscribed in a circle so that the midpoints of the sides of the polygon are concyclic on a fixed circle. Then the centroid of equally weighted particles placed at the vertices of the variable polygon remains invariant.*

The writer is unaware of any extension of Weill's theorem to the situation where the two circles are replaced by any two conics. However, since centroids are preserved under parallel projections, we may state

THEOREM 9. *If a variable n-gon is inscribed in a conic and circumscribed about a homothetic conic, then the centroid of a system of equally weighted particles placed at the points of contact of the n-gon's sides with the latter conic remains invariant.*

## THE EQUICENTER

Let us be given $n$ weighted particles $A_1, \cdots, A_{n-1}, A_n$ and let $A_1, \cdots, A_{n-1}$ be displaced in space to new positions $B_1, \cdots, B_{n-1}$. It is easy to show that in general there exists a unique position $B_n$ such that if $A_n$ is displaced to $B_n$ the centroid of the $n$ $B$-particles will coincide with the centroid of the $n$ $A$-particles. Allied to this is the interesting

**PROBLEM.** *What masses must be assigned to particles located at $A_1$, $A_2$, $A_3$ so that when displaced in the plane of the triangle $A_1A_2A_3$ to $B_1$, $B_2$, $B_3$ respectively, the centroid $M$ of the three particles will remain invariant, and how do we locate $M$?*

The point $M$ has been named the *equicenter* of the two triangles $A_1A_2A_3$ and $B_1B_2B_3$; it is the point having the same barycentric coordinates with respect to the two triangles.

Toward solving the above problem we first establish the

**LEMMA.** *Given two line segments $A_1A_2$ and $B_1B_2$, intersecting (produced if necessary) in $D_3$. Let $RD_3$ be any line through $D_3$ and let $P$ be a variable point on $RD_3$. Then*

$$\Delta A_1PA_2/\Delta B_1PB_2 = constant.$$

Designate the perpendiculars from $P$ on $A_1A_2$ and $B_1B_2$ by $a$ and $b$. Then

$$\Delta A_1PA_2/\Delta B_1PB_2 = a(A_1A_2)/b(B_1B_2) = \text{constant},$$

since $a/b$ is constant for points on $RD_3$.

Recalling that triangle $A_1'A_2'A_3'$ is said to be *anticomplementary* to triangle $A_1A_2A_3$ if $A_1, A_2, A_3$ are the midpoints of $A_2'A_3', A_3'A_1', A_1'A_2'$ respectively, we may now state and prove

**THEOREM 10.** *Let $A_1'A_2'A_3'$ and $B_1'B_2'B_3'$ be the anticomplementary triangles of two coplanar triangles $A_1A_2A_3$ and $B_1B_2B_3$ respectively. Let the corresponding sides of $A_1A_2A_3, B_1B_2B_3$ intersect in $D_1, D_2, D_3$, and those of $A_1'A_2'A_3', B_1'B_2'B_3'$ intersect in $E_1, E_2, E_3$. Then $D_1E_1, D_2E_2, D_3E_3$ are concurrent in $M$, the equicenter of triangles $A_1A_2A_3, B_1B_2B_3$. The masses at $A_1, A_2, A_3$ must be proportional to the signed areas of the triangles $A_2MA_3, A_3MA_1, A_1MA_2$.*

By the lemma we see that $D_1E_1$ is the locus of points $P$ such that

$$\Delta A_2PA_3/\Delta B_2PB_3 = \Delta A_2E_1A_3/\Delta B_2E_1B_3 = \Delta A_2A_1A_3/\Delta B_2B_1B_3 = k, \text{ say.}$$

Similarly $D_2E_2$ and $D_3E_3$ are loci of points such that

$$\Delta A_3PA_1/\Delta B_3PB_1 = \Delta A_1PA_2/\Delta B_1PB_2 = k.$$

This is sufficient to guarantee that $D_1E_1, D_2E_2, D_3E_3$ are concurrent at the equicenter $M$ of $A_1A_2A_3$ and $B_1B_2B_3$. The ratios of the masses that must be placed at $A_1, A_2, A_3$ now readily follows.

This theorem has been announced by E. Duporcq, *Intermédiare des Mathématiciens*, 1899, p. 98 and Sollertinski, *Mem. de la Soc. Royale des Sciences de Liège*, 3rd series, t. X, 1913.

The corresponding theorem in space, announced by J. Neuberg, *Mem. de*

*la Soc. Royale des Sciences de Liège*, 1913, *loc. cit.*, may be proved in an analogous manner. Here we define tetrahedron $A_1'A_2'A_3'A_4'$ to be *anti-complementary* to tetrahedron $A_1A_2A_3A_4$ if $A_1, A_2, A_3, A_4$ are the geometrical centroids of the faces $A_2'A_3'A_4', A_3'A_4'A_1', A_4'A_1'A_2', A_1'A_2'A_3'$ respectively. As a statement of the theorem we have

**THEOREM** 11. *If we designate by $a_1, a_2, a_3, a_4$ the lines of intersection of the corresponding faces of two tetrahedra $T_1$ and $T_2$, and by $a_1', a_2', a_3', a_4'$ the lines of intersection of the corresponding faces of the tetrahedra complementary to $T_1$ and $T_2$, then $a_1$ and $a_1'$, $a_2$ and $a_2'$, $a_3$ and $a_3'$, $a_4$ and $a_4'$ are coplanar, and the planes $(a_1, a_1'), (a_2, a_2'), (a_3, a_3'), (a_4, a_4')$ are copunctual at $M$, the equicenter of $T_1$ and $T_2$.*

Returning to the plane case we note that our theorem 2 permits us to state

**THEOREM** 12. *Let $A_2B_2$ and $A_3B_3$ intersect in $S_1$, $A_3B_3$ and $A_1B_1$ in $S_2$, $A_1B_1$ and $A_2B_2$ in $S_3$, and let $m_1, m_2, m_3$ denote the weights to be assigned to the particles at $A_1, A_2, A_3$. Then*

$$m_1 : m_2 : m_3 = S_2S_3 \,/\, A_1B_1 : S_3S_1 \,/\, A_2B_2 : S_1S_2 \,/\, A_3B_3 .$$

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# DIFFERENTIATION OF LOGARITHMS

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There are two methods of approach to logarithmic functions in calculus. The first one, used in most textbooks, bases the differentiation of logarithms on an expression for $e$ as a limit. The existence of this limit is seldom shown and is certainly not obvious at all. The second approach, not so frequently encountered, starts with a definition of the natural logarithm as an indefinite integral [1, p. 221; 2, p. 132]. It hinges on the integrability of a continuous function, a fact not very often fully understood by the student. The second treatment leads to the representation of $e$ as a limit rather easily [1, p. 279; 2, p. 147]. In most cases, we presume, complete rigor is not achieved in either method; the instructor prefers to make an assumption in order to obviate the main difficulties we have mentioned. If then something has to be assumed, we would like to offer the following approach, which makes the most of what the student already knows.

Assuming the existence of the derivative of a logarithm, we can set

$$D_x \log_a x = f_a(x).$$

We now differentiate the known relation

$$\log_a x = \log_b x \times \log_a b \quad \text{and obtain} \quad f_a(x) = f_b(x) \times \log_a b.$$

Substituting $x = 1$ we find

$$f_a(1) = f_b(1) \times \log_a b.$$

If $a$ is a given number, $\log_a b$ takes on all values between $-\infty$ and $+\infty$ for the possible values of $b$ [2, p. 135]. Thus we can find a number $b = e$ such that $f_a(1) = \log_a e$ and

$$f_e(1) = 1.$$

Restricting ourselves to logarithms to the base $e$ we differentiate another known identity, namely

$$\log_e Nx = \log_e N + \log_e x \quad \text{and have} \quad Nf_e(Nx) = f_e(x).$$

Substituting $x = 1$ it is seen that

$$f_e(N) = 1/N.$$

This now leads into the well established pattern of the first approach.

We conclude with an observation for those who prefer the second approach in its rigorous form. In that case our method could still be retained for its heuristic value. It tells us, what the derivative of the natural logarithm must be, if it exists.

## REFERENCES

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# ALGEBRAS BASED ON LINEAR FUNCTIONS*

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*Introduction*. In studying the structures of certain algebras, one often finds it convenient to construct an auxiliary algebra by defining a new multiplication among the elements of the original algebra in terms of the original multiplication. In this paper we shall give such a construction in terms of linear functions and shall determine the conditions on these linear functions in order that the resulting algebras shall be of certain types.

*Preliminary remarks*. First the definitions of the various types of algebras we shall consider are given. An algebra $A$ is said to be *associative* if $(xy)z = x(yz)$ for every $x$, $y$, and $z$ of $A$. A *flexible* algebra is one for which $(xy)x = x(yx)$ for every $x$ and $y$ of $A$. In order to define a power associative algebra, define the powers of an element $x$ recursively by $x^m = x^{m-1}x$ $(m = 2, 3, \cdots)$. Then $A$ is said to be *power associative* if $x^{m+n} = x^m x^n$ for all positive integers $m$ and $n$.

Given an algebra $A$, there exists an attached algebra $A^{(-)}$ which is defined to be the same vector space as $A$ but with products defined by $x \cdot y = xy - yx$ where $xy$ is the product in $A$. If $A^{(-)}$ is a Lie algebra, $A$ is said to be *Lie Admissible* [1][1].

Consider first the case where $A$ is power associative and $A^{(-)}$ is the simple Lie algebra of dimension three with basis $u_1$, $u_2$, $u_3$ and multiplication table $u_1 \cdot u_2 = u_3$, $u_2 \cdot u_3 = u_1$, $u_3 \cdot u_1 = u_2$. It will be assumed throughout that the characteristic of the base field is not two. Let $x = \alpha u_1 + \beta u_2 + \gamma u_3$ be a non-zero element of $A$ and let $x^2 = \lambda u_1 + \mu u_2 + \delta u_3$. Then since $x^2 x = x x^2$, we have $x^2 \cdot x = 0$ and direct multiplication gives $x^2 \cdot x = (\mu\gamma - \delta\beta)u_1 + (\delta\alpha - \lambda\gamma)u_2 + (\lambda\beta - \mu\alpha)u_3 = 0$. This implies $\mu\gamma = \delta\beta$, $\lambda\gamma = \delta\alpha$, and $\lambda\beta = \mu\alpha$. At least one of the values $\alpha$, $\beta$, or $\gamma$ is distinct from zero, so let us assume that $\gamma \neq 0$. Then we have $\lambda = \frac{\delta\alpha}{\gamma}$, $\mu = \frac{\delta\beta}{\gamma}$, and $x^2 = \frac{\delta}{\gamma}[\alpha u_1 + \beta u_2 + \gamma u_3] = \frac{\delta}{\gamma}x$. Since $x$ was an arbitrary non-zero element of $A$, we have $x^2 = f(x)x$ for every element $x$ of $A$, where $f(x)$ is in the base field.

**LEMMA** 1. *The function $f$ is linear.*

PROOF. First let $x^2 = f(x)x$. Then $(\alpha x)^2 = \alpha^2 x^2 = \alpha^2 f(x)x = \alpha f(x)(\alpha x)$.

* Presented to the American Mathematical Society; April, 1953.
1. Numbers in brackets refer to the bibliography at the end of the paper.

Hence $f(ax)=af(x)$. We have $(x+y)^2=f(x+y)(x+y)=f(x)x+f(y)y+xy+yx$. Solving for $xy+yx$ yields $xy+yx=[f(x+y)-f(x)]x+[f(x+y)-f(y)]y$. Also, $(x+x+y)^2=f(x+x+y)(x+x+y)=f(x)x+f(x+y)(x+y)+x(x+y)+(x+y)x$. Solving for $x(x+y)+(x+y)x$ yields $x(x+y)+(x+y)x=[f(x+x+y)-f(x)]x+[f(x+x+y)-f(x+y)](x+y)=2x^2+xy+yx=2f(x)x+[f(x+y)-f(x)]x+[f(x+y)-f(y)]y$. But then $x[2f(x+x+y)-2f(x)-2f(x+y)]+y[f(x+x+y)-2f(x+y)+f(y)]=0$. We may assume $x$ and $y$ to be linearly independent; otherwise, the lemma is already proved. Then

$$2f(x+x+y)-2f(x+y)=2f(x),$$

$$f(x+x+y)-2f(x+y)=-f(y),$$

$$f(x+y)=f(x)+f(y)$$

as desired.

***THEOREM* 1.** *If $A$ is power associative, and $A^{(-)}$ is the simple Lie algebra of dimension three, then multiplication in $A$ is given by*

$$xy=\tfrac{1}{2}[f(x)y+f(y)x+x\cdot y]. \tag{1}$$

***PROOF.*** By the lemma we have $(x+y)^2=[f(x)+f(y)](x+y)$. Then $(x+y)^2=f(x)x+f(y)y+f(x)y+f(y)x=x^2+xy+yx+y^2=f(x)x+f(y)y+xy+yx$. It follows that $xy+yx=f(x)y+f(y)x$. But then $xy=\tfrac{1}{2}(xy+yx)+\tfrac{1}{2}(xy-yx)=\tfrac{1}{2}[f(x)y+f(y)x+x\cdot y]$.

This theorem suggests the following construction. Let $A$ be a given algebra with multiplication $xy$ and let $A'$ be the algebra derived from $A$ by redefining multiplication of elements in terms of a linear function $f$ by

$$x\cdot y=f(x)y+f(y)x+xy. \tag{2}$$

*The associative case.* We assume that $A$ is an associative algebra of dimension $n>1$ and that $A'$ is defined as in (2). If $A'$ is to be associative we must have $(x\cdot y)\cdot z=x\cdot(y\cdot z)$, $[f(y)x+f(x)y+xy]\cdot z=x\cdot[f(z)y+f(y)z+yz]$, $f(y)[f(z)x+f(x)z+xz]+f(x)[f(z)y+f(y)z+yz]+f(z)(xy)+f(xy)z+(xy)z=f(z)[f(y)x+f(x)y+xy]+f(y)[f(z)x+f(x)z+xz]+f(yz)x+f(x)(yz)+x(yz)$,

$$[f(x)f(y)+f(xy)]z=[f(y)f(z)+f(yz)]x. \tag{3}$$

Taking $x$ and $z$ to be linearly independent we have $f(x)f(y)=-f(xy)$ as a necessary condition for the associativity if $A'$. Conversely, this condition is also sufficient. This proves:

***THEOREM* 2.** *If $A$ is an associative algebra of dimension $n>1$, a necessary and sufficient condition for the associativity of $A'$ is*

$$f(x)f(y)=-f(xy). \tag{4}$$

In terms of a basis $u_1, \cdots, u_n$ of $A$ and the constants of multiplication $c_{ijk}$, this condition may be expressed as

$$f(u_i)f(u_j) = -\sum_k c_{ijk} f(u_k). \tag{5}$$

As an example, let $A$ be the diagonal algebra with basis $u_1, u_2, \cdots, u_n$ and $f$ be defined by $f(u_1) = -1$, $f(u_2) = f(u_3) = \cdots = f(u_n) = 0$. This function is seen to satisfy condition (5), and the resulting algebra $A'$ will therefore be associative.

Replacing $z$ by $x$ in (3) yields $f(x)f(y) + f(xy) = f(x)f(y) + f(yx)$, $f(xy) = f(yx)$. Clearly, this condition satisfies (3) with $z$ replaced by $x$. Thus we have

**COROLLARY** 1. *If $A$ is a flexible algebra of dimension $n > 1$, a necessary and sufficient condition for the flexibility of $A'$ is*

$$f(xy) = f(yx). \tag{6}$$

*The power associative case.* In the power associative case we shall restrict ourselves to the case where the base field $F$ is of characteristic zero. It is known [2] that over such a field the conditions $(x \cdot x) \cdot x = x \cdot (x \cdot x)$ and $(x \cdot x) \cdot (x \cdot x) = [(x \cdot x) \cdot x] \cdot x$ are sufficient for the power associativity of $A'$. The condition $(x \cdot x) \cdot x = x \cdot (x \cdot x)$ is equivalent to $[4f(x)^2 + f(x^2)]x + 3f(x)x^2 + x^2x = [4f(x)^2 + f(x^2)]x + 3f(x)x^2 + xx^2$ which is satisfied when $A$ is power associative. The condition $(x \cdot x) \cdot (x \cdot x) = [(x \cdot x) \cdot x] \cdot x$ yields the following:

**THEOREM** 3. *Let $A$ be a power associative algebra over a field of characteristic zero. Then a necessary and sufficient condition for the power associativity of $A'$ is*

$$[f(x^2)f(x) + f(x^3)]x - [f(x)^2 + f(x^2)]x^2 = 0. \tag{7}$$

If $x^2$ is not a scalar multiple of $x$, then (7) is equivalent to

$$f(x)^2 + f(x^2) = 0 \tag{8}$$

and

$$f(x^2)f(x) + f(x^3) = 0. \tag{9}$$

A linearization of (8) gives

$$f(x)f(y) = -\tfrac{1}{2}[f(xy) + f(yx)], \tag{10}$$

and substituting $y = x^2$ in (10) yields (9). Thus either (10) or the anticommutativity of $A$ constitutes a sufficient condition for the power associativity of $A'$.

As an example we take a power associative algebra $B$ and adjoin an

identity element e. Then the algebra obtained is $A = B + e$. Define $f(e) = -1$, $f(b) = 0$ for $b$ in $B$. This function $f$ is seen to satisfy (10) and will therefore give rise to a power associative algebra $A'$.

*The construction of simple algebras.* We shall assume now that the algebra $A$ is an anticommutative algebra over a field of characteristic zero, so that the resulting algebra $A'$ will be power associative. Let $B$ be a proper linear subspace of the algebra $A$, let $b$ be an element of $B$, and let $a$ be an element of $A$ which is not in $B$. Suppose first that $B$ is not an ideal of $A$. Since $a \cdot b - b \cdot a = 2ab$, any ideal of $A'$ is an ideal of $A$, and consequently $B$ cannot be an ideal of $A'$. Next let $B$ be an ideal of $A$. Then

$$f(b)a = a \cdot b - f(a)b - ab. \tag{11}$$

The assertion that $B$ is an ideal of $A'$ would imply that the right hand member of (11) is in $B$. This would lead to a contradiction, however, provided $f(b) \neq 0$. Thus the construction of simple algebras depends upon the question of whether it is possible to define a function $f$ on $A$ such that every non-zero ideal $B$ of $A$ contains at least one element $b$ for which $f(b) \neq 0$. When $A$ is a simple algebra or a direct sum of simple algebras, this is clearly possible. In the general case, however, we leave this question open.

## REFERENCES

1. A. A. Albert, *Power Associative Rings*, Trans. Amer. Math. Soc. vol. 64, no. 3 (1948), pp. 552-593.

2. A. A. Albert, *On the Power Associativity of Rings*, Summa Braziliensis Mathematicae vol. 2 (1948), pp. 21-33.

DePaul University

# COLLEGIATE ARTICLES

Graduate training not required for reading.

# HARMONIC POINTS AND LOCI CONNECTED WITH THE FRÉGIER THEOREM

Sister Clotilda Spezia

1. *Introduction*. The Frégier Theorem [1]* has been the occasion of some interesting geometric problems, one of which, the subject of this note, reads as follows: If $P$ is any fixed point on a given conic, $F$ the Frégier point of $P$, and $C$ the center of curvature corresponding to $P$ on the conic, find the loci of the point $Q$ harmonic with respect to the points $P,F,C$. [2] In this study we first find the coordinates of the point $Q$ harmonic to the points $P,F,C$ relative to the parabola, the ellipse, and the hyperbola; we then determine the loci of $Q$ and analyze them.

2. *Harmonic Points*. 3 When three collinear points $A(x_1,y_1)$, $B(x_2,y_2)$, and $C(x_3,y_3)$ are given, a fourth point $D(x,y)$-the harmonic conjugate-may take different positions, giving rise to three distinct cases [4] where $D$ may have the following sets of coordinates:

$$(1)\qquad x = \frac{x_3(x_1 - x_2) + x_1(x_3 - x_2)}{(x_3 - x_2) + (x_1 - x_2)}, \qquad y = \frac{y_3(y_1 - y_2) + y_1(y_3 - y_2)}{(y_3 - y_2) + (y_1 - y_2)};$$

$$(2)\qquad x = \frac{x_2(x_1 - x_3) + x_1(x_2 - x_3)}{(x_2 - x_3) + (x_1 - x_3)}, \qquad y = \frac{y_2(y_1 - y_3) + y_1(y_2 - y_3)}{(y_2 - y_3) + (y_1 - y_3)};$$

*The numbers in [ ] refer to references given at the end of this material.

and

$$x = \frac{x_2(x_3 - x_1) + x_3(x_2 - x_1)}{(x_2 - x_1) + (x_3 - x_1)}$$

(3)

$$y = \frac{y_2(y_3 - y_1) + y_3(y_2 - y_1)}{(y_2 - y_1) + (y_3 - y_1)}$$

3. *The Parabola.* If $P(x_1,y_1)$ lies on

$$y^2 = 4ax,$$

the coordinates of $F$, the Frégier point relative to $P(x_1,y_1)$, are

$$x = x_1 + 4a, \qquad y = -y_1,$$

while the coordinates [5] of $C$, the center of curvature corrrsponding to $P(x_1,y_1)$, are

$$x = 3x_1 + 2a, \qquad y = \frac{-y_1^3}{4a^2}.$$

It is not difficult to show that $P$, $F$, and $C$ are collinear on the normal to the parabola at $P(x_1,y_1)$. We now find that the coordinates of the point $Q$ harmonic to $P,F,C$ for the three cases are

$$x = \frac{x_1^2 - 7ax_1 - 4a^2}{x_1 - 3a},$$

$$y = \frac{-3y_1^3 + 4a^2y_1}{-y_1^2 + 12a^2},$$

$$x = \frac{x_1^2 + 11ax_1 + 8a^2}{x_1 + 3a};$$

$$y = \frac{3y_1^3 + 4a^2y_1}{-y_1^2 - 12a^2};$$

and

$$x = \frac{x_1^2 + 2ax_1 + 2a^2}{x_1},$$

$$y = \frac{-4a^2}{y_1}.$$

The given coordinates of $Q$ are likewise the parametric equations of the loci of $Q$. By eliminating the parameter $x_1$, the rectangular equations of the loci of $Q$ are

(1) (Fig. 1),

$$y^4 - 3x^2y^2 + 10\sigma xy^2 - 683\sigma^2y^2 + 108\sigma x^3 - 648\sigma^2x^2 + 1260\sigma^3x - 784\sigma^4 = 0,$$

(2) (Fig. 2),

$$y^4 + 3x^2y^2 - 14\sigma xy^2 + 691\sigma^2y^2 - 108\sigma x^3 + 648\sigma^2x^2 - 1260\sigma^3x + 800\sigma^4 = 0,$$

and

(3) (Fig. 3), $$y^4 + 4\sigma^2y^2 - 2\sigma xy^2 + 8\sigma^4 = 0.$$

A brief analysis of each figure follows. As a point $P(x_1, y_1)$ traces the parabola from $+\infty$ through the points indicated on the figure, a point $Q(x, y)$ traces the $Q$-locus through points correspondingly named but primed. Due to the symmetry of both the $P$-locus and the $Q$-locus with respect to the $x$-axis, the second helf of each locus is traced similarly.

In Fig. 1, the $Q$-locus has nodes at $A'$, $D'$ and $E'$; a minimum value

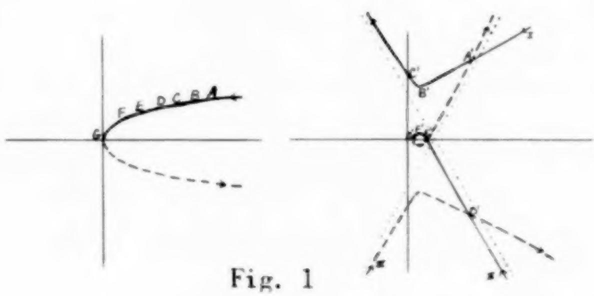


Fig. 1

at $B'$; a maximum value at $F'$; and oblique asymptotes whose equations are

$$3y = \pm\, 3\sqrt{3}\, x \mp 23\sqrt{3}\,\sigma.$$

In Fig. 2, there is on the $Q$-locus a conjugate point $B$ which was introduced as a value of the rectangular equations of the $Q$-locus when the

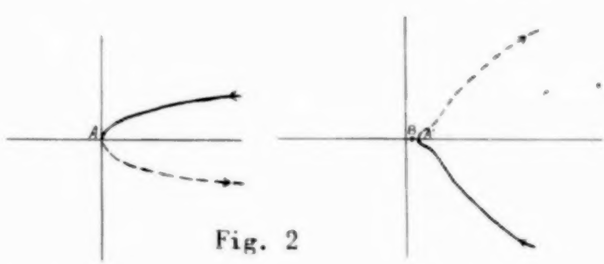


Fig. 2

parametric equations were converted to rectangular. There is no point on the $P$-locus corresponding to point $B$ on the $Q$-locus.

In Fig. 3, the $x$-axis is a horizontal asymptote to the $Q$-locus.


Fig. 3

4. *The Ellipse.* If $P(x_1, y_1)$ lies on

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

then the coordinates of the Frégier point $F$ are

$$x = \frac{x_1(a^2 - b^2)}{a^2 + b^2},$$

$$y = \frac{-y_1(a^2 - b^2)}{a^2 + b^2},$$

while those of the center of curvature $C$ are

$$x = \frac{x_1^3(a^2 - b^2)}{a^4},$$

$$y = \frac{-y_1^3(a^2 - b^2)}{b^4}.$$

Again, $P$, $F$, and $C$ are collinear on the normal to the ellipse. The coordinates of the point $G$ harmonic to $P, F, C$ for the three cases are

$$x = \frac{a^4x_1^3 - 3b^4x_1^3 + 2a^2b^2x_1^3 - a^6x_1 + a^4b^2x_1}{a^4x_1^2 - b^4x_1^2 - a^6 + 3a^4b^2},$$

(Fig. 4)

$$y = \frac{-3a^4y_1^3 + b^4y_1^3 + 2a^2b^2y_1^3 - b^6y_1 + a^2b^4y_1}{-a^4y_1^2 + b^4y_1^2 - b^6 + 3a^2b^4};$$

$$x = \frac{a^4x_1^3 + 3b^4x_1^3 - 4a^2b^2x_1^3 - a^6x_1 + a^4b^2x_1}{a^4x_1^2 - b^4x_1^2 - a^6 - 3a^4b^2},$$

$$y = \frac{3a^4y_1^3 + b^4y_1^3 - 4a^2b^2y_1^3 - b^6y_1 + a^2b^4y_1}{-a^4y_1^2 + b^4y_1^2 - b^6 - 3a^2b^4}; \qquad \text{(Fig. 5)}$$

and

$$x = \frac{-a^4x_1^3 + a^2b^2x_1^3 + a^6x_1 - a^4b^2x_1}{-a^4x_1^2 + b^4x_1^2 + a^6},$$

$$y = \frac{-b^4y_1^3 + a^2b^2y_1^3 + b^6y_1 - a^2b^4y_1}{a^4y_1^2 - b^4y_1^2 + b^6}. \qquad \text{(Fig. 6)}$$

The general appearance of the $Q$-loci (Figs. 4,5,6), analyzed from their parametric equations, follow. As both the $P$-locus and the $Q$-loci are symmetrical with respect to the axis and the origin, only one half of each locus is shown.

In Fig. 4, the $Q$-locus has a maximum value at $D'$; a minimum value at $A'$;

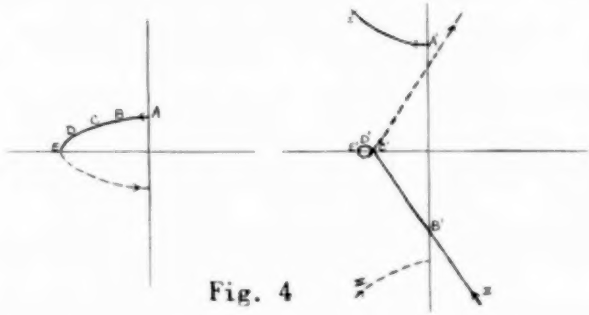



Fig. 4

a vertical tangent at $E'$; and horizontal tangents at $A'$ and $D'$.

In Fig. 5, there is a vertical tangent to the $Q$-locus at $B'$; a horizontal tangent at $A'$.

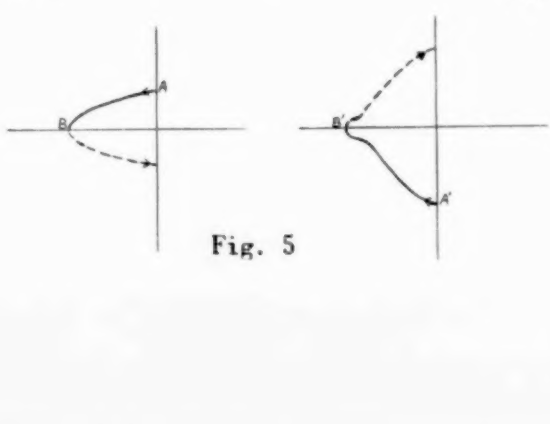



Fig. 5

In Fig. 6, the $Q$-locus has a minimum value at $C'$; vertical tangents at $A'(D')$ and $B'$; horizontal tangents at $A'(D')$ and $C'$.

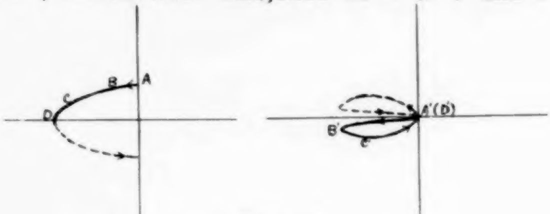


Fig. 6

5. *The Hyperbola.* If the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the coordinates of the Frégier point $F$, the center of curvature $C$, and the harmonic point $Q$ differ from those relative to the ellipse only in the sign of $b^2$.

The general appearance of the $Q$-loci, analyzed from their parametric equations for all three cases, follow in the sketches and analyses of Figs. 7,8,9. Again, as both the $P$-locus and the $Q$-loci are symmetrical with respect to the axes and the origin, only one half of each locus is traced.

In Fig. 7, the $Q$-locus has a minimum value at $A'$; a maximum value at

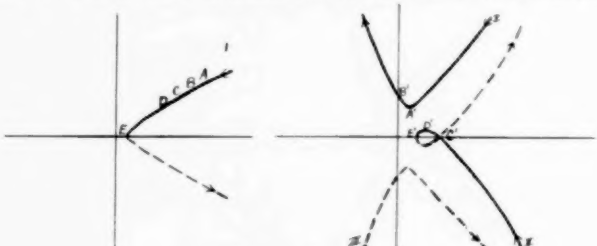


Fig. 7

$D'$; a vertical tangent at $E'$; and horizontal tangents at $A'$ and $D'$.

In Fig. 8, there is a vertical tangent to the $Q$-locus at $A'$.

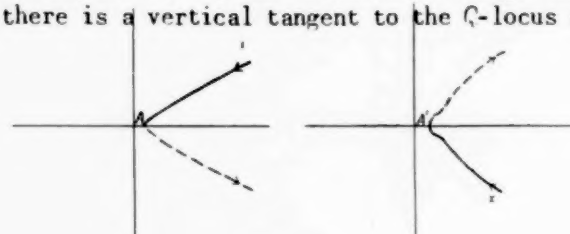


Fig. 8

In Fig. 9, the $Q$-locus has a maximum value at $A'$; vertical tangents

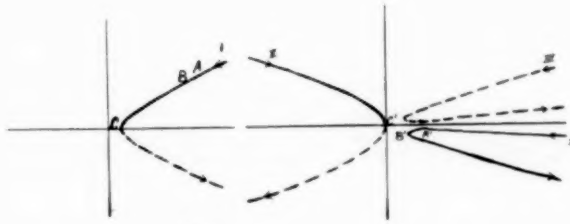


Fig. 9

at $C'$ and $B'$; and a horizontal tangent at $A'$.

Finally, it can easily be shown that the loci of $F$, $C$, and $Q$ - relative to the parabola, the ellipse, and the hyperbola - have common points of tangency.

[1] Frégier's work appeared in Gergonne's *Annales de Mathematique*, t. VI, 1816, pp. 229-241, pp. 321-326; and t. VII, 1816, pp. 95-99.

[2] The *Revue de Mathematique Speciales* (Vol. 17-18, 1920-1921) carried this problem but only gave an analysis of the $Q$-loci in parametric form relative to the parabola. pp. 140, 469-471.

[3] Nathan Altshiller-Court. *College Geometry*. Chicago: Johnson Publishing Company, 1925. pp. 134-135.

[4] Sister Clotilda Spezia. *Harmonic Points and Loci Connected with the Frégier Theorem*. Master's Thesis, St. Louis University, 1951. pp. 12-25.

[5] W. A. Granville, P. Smith, and W. R. Langley. *Elements of the Differential and Integral Calculus*. Chicago: Ginn and Co., 1929. p. 157.

St. Louis University

*Announcement to Publishers and Authors of Books on Mathematics*

The increasing number of books coming to us for reviewing has made it impossible to secure and publish the reviews, in the usual way, within a reasonable length of time after the publication of the books. Hence we have devised a new procedure which we believe will remove some of the present hurdles and still give our readers the essential characteristics of the books.

From now on we will publish two types of announcements if furnished by the publishers and submitted to us within a year before or after the publication of the book. These announcements must be signed by their authors, require no hand-setting and be subject to the same editorial criticism as other material that we publish.

1) Announcements of texts on standard courses, limited to 120 words.
2) Longer releases on books in new fields or very unusual treatments of established courses, limited to 500 words.

The objection has been raised that we will not get negative criticism. The answer is that negative criticism is rare under the old system and is almost never of any serious value either to the publisher or prospective user. A further objection has been that we cannot get an unprejudiced description of a book from one who has a financial interest in it. The answer to this is that we want the novel and interesting features of the books emphasized and believe that this will be done honestly.

Reviews on hand or being prepared will be published as fast as possible. Books currently received are being returned to their publishers.

Books not eligable to the above announcements may be advertised at 75¢ per line, without hand-setting. Such ads may be run in any space which will not throw them into reprints of articles.

We hope we will receive your cooperation in this new plan for handling reviews.

Glenn James, Mg. Ed.

# TEACHING OF MATHEMATICS

*Edited by*

Joseph Seidlin and C. N. Shuster

**This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.**

## INEQUALITIES*

Richard Bellman

It has been said that mathematics is the science of tautology, which is to say that mathematicians spend their time proving that equal quantities are equal. This statement is wrong on two counts: In the first place, mathematics is not a science, it is an art; in the second place, it is fundamentally the study of inequalities rather than equalities.

I would like today to discuss a number of the basic inequalities of analysis, presenting first an algebraic proof of the inequality between the arithmetic and geometric means, and then a most elegant geometric technique due to Young. In passing we will observe how the theory of inequalities may be used to supplant the calculus in many common types of maximization and minimization problems. Finally, I shall show how Young's inequality leads naturally to Holder's inequality, and Holder's inequality to Minkowski's.

Since it has become unfashionable in educational circles to pose problems in the spirit of a spelling bee, but rather to motivate the student by relating the problem to our everyday pursuits, we shall consider the following question which is perhaps typical of the way in which the theory of inequalities can enter into our ordinary pursuits.

A football player of some renown, having gone into stocks and bonds and made a substantial score there also, stipulated in his will that his coffin be enclosed in a giant football. His executors, of an economical turn of mind, were confronted with the problem of finding the dimensions of the smallest football which would meet the terms of the will.

Indulging in the usual mathematical license, we may consider the football to be an ellipsoid and reduce the problem to that of finding the coffin of maximum volume which will fit into a given ellipsoid.

Taking the ellipsoid to have the equation $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, we see the analytical equivalent of the practical problem above is that of finding the maximum of $v = 8xyz$ subject to the above constraint on $x$, $y$ and $z$.

*Delivered before a Teachers Conference at Univ. of Calif. at L. A.

Let us first observe that we can simplify by observing that $v$ and $v^2/a^2b^2c^2$ are maximized simultaneously. Hence replacing $x^2/a^2$ by $u$, $y^2/b^2$ by $v$, $z^2/w^2$ by $w$, the problem reduces to maximizing $uvw$ subject to $u+v+w=1$, $u,v,w \geq 0$. It is intuitive now that the symmetric point $u=v=w=1/3$ should play a distinguished role, as either a minimum or a maximum. Since it is clearly not a minimum it follows that it furnishes the desired maximum.

§2. *An algebraic Approach.*

In order to prove this, in a purely algebraic fashion without the aid of calculus, we shall derive a general inequality connecting the arithmetic mean of $n$ positive quantities, $(a_1+a_2+\cdots+a_n)/n$, and the geometric mean, $\sqrt[n]{a_1a_2\cdots a_n}$, namely

$$\sqrt[n]{a_1a_2\cdots a_n} \leq \frac{a_1+a_2+\cdots+a_n}{n} \tag{1}$$

with equality occurring only if

$$a_1 = a_2 = \cdots = a_n . \tag{2}$$

There are literally hundreds of proofs of this basic inequality, many of which are actually quite different. The proof I will present is perhaps not the simplest, but it is one of the most ingenious. It is perhaps the only application of a particular form of mathematical induction and I think that it will be interesting for that reason.

The proof begins in a very simple manner. The most basic inequality, which I must confess is a tautology, is that a non-negative number is greater than or equal to zero. The simplest non-negative number, and one which is invariably non-negative, is a square of a number. Taking, for our own purposes, (and this is where ingenuity enters), the number $a-b$ and squaring it, we have the inequality

$$(a-b)^2 \geq 0 \tag{3}$$

Multiplying out and transposing, we have the well-known inequality,

$$\frac{a^2+b^2}{2} \geq ab \tag{4}$$

together with the important addition that equality can occur if and only if $a=b$. Setting $a^2=a_1$, $b^2=b_1$, we have the well-known result that the arithmetic mean of 2 positive quantities is greater than or equal to their geometric mean.

Let us now replace $a_1$ by $(a_1+a_2)/2$ and $b_1$ by $(a_3+a_4)/2$ obtaining

$$\frac{a_1+a_2+a_3+a_4}{4} \geq \sqrt{\left(\frac{a_1+a_2}{2}\right)\left(\frac{a_3+a_4}{2}\right)} \tag{5}$$

and apply the separate inequalities

$$(6) \qquad \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}, \qquad \frac{a_3 + a_4}{2} \geq \sqrt{a_3 a_4}$$

obtaining

$$(7) \qquad \frac{a_1 + a_2 + a_3 + a_4}{4} \geq \sqrt{a_1 a_2 a_3 a_4}$$

Retracing our steps we see that we still have the important fact that equality can occur only if $a_1 = a_2 = a_3 = a_4$.

Continuing in this way we obtain, for $n$ a power of two, the inequality

$$(8) \qquad \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

with equality occurring only if all the variables are equal.

We still cannot apply this to our problem since 3 is not a power of two. We want to show that the same inequality holds for $n = 3$. Once more we require some ingenuity. Let us take the case $n = 4$, and set

$$(9) \qquad a_1 = a_1, \quad a_2 = a_2, \quad a_3 = a_3, \quad a_4 = \frac{a_1 + a_2 + a_3}{3}.$$

The resulting inequality is

$$(10) \qquad \left(\frac{a_1 + a_2 + a_3}{3}\right) \geq \sqrt[4]{a_1 a_2 a_3 \left(\frac{a_1 + a_2 + a_3}{3}\right)}$$

which simplifies to

$$(11) \qquad \frac{a_1 + a_2 + a_3}{3} \geq \sqrt{a_1 a_2 a_3}$$

the desired inequality for three. Retracing our steps we see that equality can occur only if we have $a_1 = a_2 = a_3$.

This technique is perfectly general and yields the inequality for $n - 1$ whenever it has been established for $n$. Since we have established it for the integers 2, 4, 8, etc., we see that induction yields it for all $n$. Observe, however, that this is a backward induction rather than the usual forward induction.

Turning to our original problem we see that

$$(12) \qquad 1/3 = \frac{u + v + w}{3} \geq \sqrt[3]{uvw}$$

unless $u = v = w = 1/3$. In terms of the original variables, $x$, $y$, and $z$ this yields

(13) $$x = a/\sqrt{3}, \quad y = b/\sqrt{3}, \quad z = c/\sqrt{3}$$

as the solution of our maximization problem.

Returning to (8), and taking $n = 10$, we obtain an interesting inequality by grouping the variables as follows

(13) $$\begin{aligned} a_1 &= a_2 = b_1 \\ a_3 &= a_4 = a_5 = b_2 \\ a_6 &= a_7 = a_8 = a_9 = a_{10} = b_3 \end{aligned}$$

The result is

(14) $$\frac{2b_1}{10} + \frac{3b_3}{10} + \frac{5b_2}{10} \geq b_1^{2/10} b_2^{3/10} b_3^{5/10}$$

which is a particular case of the general inequality

(15) $$\left[\frac{n_1 b_1 + n_2 b_2 + \cdots + n_k b_k}{n_1 + n_2 + \cdots + n_k}\right]^{(n_1 + n_2 + \cdots + n_k)} \geq b_1^{n_1} b_2^{n_2} \cdots b_k^{n_k}$$

where the $n_1, n_2, \cdots, n_k$ are positive integers, $b_1, b_2, \cdots, b_k$ are positive, and equality occurs only if $b_1 = b_2 = \cdots = b_k$.

The limiting form of (15), namely

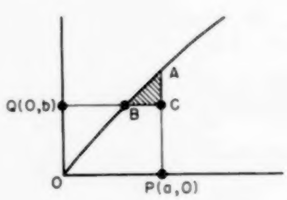
(16) $$\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_k b_k \geq b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_k^{\alpha_k}$$

where $\alpha_i \geq 0$, $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$ is also valid, but is, of course, no longer a purely algebraic theorem.

I leave it to the reader to use the above inequality to determine the maximum of $xyz$ subject to $x^2 + y^3 + z^5 = 1$, $x, y, z \geq 0$.

§3. *A Geometric Approach.*

Let us now turn to an alternate approach to the theory of inequalities. Let $f(x)$ be a monotone-increasing function with $f(0) = 0$ and consider the diagram below




The area under $OAP$ is given by $\int_0^a f(x)dx$, while that under $OQB$ is given by $\int_0^b f^{-1}(x)dx$ (where $f^{-1}(x)$ denotes the inverse function) clearly the

sum of these two areas is greater than or equal to that of the rectangle $OQCP$, with equality occurring if and only if $b = f(a)$. Writing this statement in analytical terms we obtain the inequality of Young,

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab, \tag{17}$$

for $a,b \geq 0$.

Taking $f(x) = x$, we obtain (4) above. Taking $f(x) = x^{p-1}$, where $p > 1$, we obtain

$$a^p/p + b^{p'}/p' \geq ab \tag{18}$$

where $p' = p/p-1$ (note that $1/p + 1/p' = 1$), If $f(x) = \log(1+x)$, we obtain, after some simplification

$$(1+a)(\log(1+a)) - (1+a) + e^b - b \geq ab, \quad a,b \geq 0 \tag{19}$$

an inequality which is important in the theory of Fourier series.

## §4. *Holder's Inequality.*

Let us now show how (19) yields one of the most useful inequalities of analysis, the classical inequality of Holder,

$$(a_1^p + a_2^p + \cdots + a_n^p)^{1/p} (b_1^{p'} + b_2^{p'} + \cdots + b_n^{p'})^{1/p'} \geq (a_1b_1 + \cdots + a_nb_n) \tag{20}$$

for $a_i, b_i \geq 0$, $p > 1$.

In (18) set successively

$$\begin{aligned} a &= a_i/(a_1^p + a_2^p + \cdots + a_n^p)^{1/p} \\ b &= b_i/(b_1^{p'} + b_2^{p'} + \cdots + b_n^{p'})^{1/p'} \end{aligned} \tag{21}$$

and add. On the right side we obtain

$$\frac{a_1b_1 + \cdots + a_nb_n}{(a_1^p + a_2^p + \cdots + a_n^p)^{1/p} (b_1^{p'} + b_2^{p'} + \cdots + b_n^{p'})^{1/p'}}. \tag{22}$$

while on the left-hand side we obtain

$$\frac{1}{p}\left[\frac{a_1^p + a_2^p + \cdots + a_n^p}{a_1^p + a_2^p + \cdots + a_n^p}\right] + \frac{1}{p'}\left[\frac{b_1^{p'} + b_2^{p'} + \cdots + b_n^{p'}}{b_1^{p'} + b_2^{p'} + \cdots + b_n^{p'}}\right] = \frac{1}{p} + \frac{1}{p'} = 1 \tag{23}$$

This yields Holder's inequality and shows, using the condition of equality in (18) that equality holds in (20) if and only if $b_i = a_i^{p-1}$ for $i = 1, 2, \cdots, n$.

An alternate form of Hölder's inequality which we shall use below to derive Minkowski's inequality is

$$(24) \qquad \max_B \,(a_1b_1 + \cdots + a_nb_n) = (a_1^p + a_2^p + \cdots + a_n^p)^{1/p}$$

where $B$ represents the domain: $b_i \geq 0$, $b^{p'} + b^{p'} + \cdots + b_n^{p'} = 1$, and $q_i \geq 0$. This follows from the observation that the righthand side is an upper bound according to Holder's inequality and is attained for $b_i = a_i^{p-1}$.

§5. *Minkowski's Inequality.*

Using (25) we have, for $x_k, y_k \geq 0$,

$$[(x_1 + y_1)^p + (x_2 + y_2)^p + \cdots + (x_n + y_n)^p]^{1/p}$$

$$= \max_B \,[(x_1 + y_1)b_1 + (x_2 + y_2)b_2 + \cdots + (x_n + y_n)b_n]$$

$$= \max_B \,[(x_1b_1 + x_2b_2 + \cdots + x_nb_n) + (y_1b_1 + y_2b_2 + \cdots + y_nb_n)]$$

$$= \max_B \,(x_1b_1 + x_2b_2 + \cdots + x_nb_n) + \max_B \,(y_1b_1 + y_2b_2 + \cdots + y_nb_n)$$

$$= (x_1^p + x_2^p + \cdots + x_n^p)^{1/p} + (y_1^p + y_2^p + \cdots + y_n^p)^{1/p}$$

We have thus established the classical inequality of Minkowski, which is for $p = p' = 2$, the famous "triangle inequality" of Euclid which states that the sum of two sides of a triangle is greater than the third. Put another way, a straight line is the shortest distance between two points.

§6. *In Conclusion.*

In presenting these two approaches to the theory of inequalities, I have neglected perhaps the most powerful approach, that based upon the concept of a convex function. This, however, deserves its own presentation.

For those interested in learning more about inequalities, I refer to that fascinating book by Hardy, Littlewood and Polya entitled quite simply "Inequalities."

Rand Corporation

# PROBLEMS AND QUESTIONS

*Edited by*

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

## PROPOSALS

**208.** *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Evaluate the following trigonometric expressions without using numerical tables:

$$A = \cos 5^\circ \cos 10^\circ \cos 15^\circ \cdots \cos 75^\circ \cos 80^\circ \cos 85^\circ,$$

$$B = \cos 1^\circ \cos 3^\circ \cos 5^\circ \cdots \cos 85^\circ \cos 87^\circ \cos 89^\circ,$$

$$C = \cos 4^\circ \cos 8^\circ \cos 12^\circ \cdots \cos 80^\circ \cos 84^\circ \cos 88^\circ.$$

**209.** *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Show that $F(x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{a^{n+1} + y}$ is symmetric in $x$ and $y$.

**210.** *Proposed by Barney Bissinger, Lebanon Valley College.*

A differential equation yielded the solution:

$$y = \frac{g^{c^t} - g}{g^t - 1} \quad \text{for } t > 0.$$

How should the exponents be interpreted to make y finite as $t \to 0$?

**211.** *Proposed by J. Lambeck, McGill University and L. Moser, University of Alberta.*

Prove that $\sum_{i=1}^{n^2+n} \{\sqrt{i}\} = 2 \sum_{i=1}^{n} i^2$ where $\{x\}$ denotes the integer closest to $x$.

**212.** *Proposed by F. J. Duarte, Caracas, Venezuela.*

Prove that the number $\dfrac{1+[1+(10^{10}-1)^{99989}]\,(10^{999890}-1)}{99991}$ is an integer.

**213.** *Proposed by John M. Howell, Los Angeles City College.*

$A$ has $a$ dollars and $B$ bas $b$ dollars. They play a series of games in which $A$ has a probability $p$ of winning, probability $q$ of losing and probability $t$ of tying in each game. If they wager one dollar on each game, what is the probability of $A$ winning all of the money?

**214.** *Proposed by N. A. Court, University of Oklahoma.*

The radical circle of two given orthogonal circles is the circle of similitude of the two circles of antisimilitude of the given circles.

## SOLUTIONS

### *Late Solutions*

**182.** *M. S. Klamkin, Polytechnic Institute of Brooklyn.*

**184, 186.** *M. S. Klamkin, Polytechnic Institute of Brooklyn; E. P. Starke, Rutgers University.*

### A Minimal Tetrahedron

**157.** [January 1953] *Proposed by W. H. Glenn Jr., Pasadena City Schools, California.*

In Woods and Bailey, *Analytic Geometry and Calculus,* Ginn (1944), page 365 problem 69 states: "Through a given point (1, 1, 2) a plane is passed which, with the coordinate planes, forms a tetrahedron of minimum volume. Find the equation of the plane." A student set up the volume as $V = abc/6$ where $a$, $b$ and $c$ are the $x$, $y$, $z$ intercepts of the plane through (1, 1, 2). He then took $\partial V/\partial a$, $\partial V/\partial b$, $\partial V/\partial c$ and evaluated them at $a = 1$, $b = 1$, $c = 2$. He then said that the plane he was seeking was:

$$(\partial V/\partial a)_p\,(x-1) + (\partial V/\partial b)_p\,(y-1) + (\partial V/\partial c)_p\,(z-2) = 0$$

where the subscript $p$ indicates that the partial derivative has been evaluated at $p(1, 1, 2)$. Show that this result is correct and that the procedure can be generalized to $p(x_1, y_1, z_1)$.

*Solution by the proposer.* Assume the general form of the plane through the point $(x_1, y_1, z_1)$ as

$$(1) \qquad A(x-x_1) + B(y-y_1) + C(z-z_1) = 0.$$

The three intercepts $a$, $b$ and $c$ are

$$a = \frac{Ax_1 + By_1 + Cz_1}{A} \qquad b = \frac{Ax_1 + By_1 + Cz_1}{B} \qquad c = \frac{Ax_1 + By_1 + Cz_1}{C}$$

The volume of the tetrahedron is

$$V = \frac{1}{6}\,\frac{(Ax_1 + By_1 + Cz_1)^3}{ABC}$$

Since (1) is homogeneous in the unknowns $A$, $B$, and $C$ it is sufficient for the condition of maximum and minimum to have:

$$\frac{\partial V}{\partial A} = 0, \qquad \frac{\partial V}{\partial B} = 0$$

The following conditions result

$$3Ax_1 - (Ax_1 + By_1 + Cz_1) = 0 \qquad 3By_1 - (Ax_1 + By_1 + Cz_1) = 0$$

and solving for $A$ and $B$, we have

$$A = \frac{Cz_1}{x_1} \qquad B = \frac{Cz_1}{y_1}$$

Therefore

$$A : B : C = \frac{Cz_1}{x_1} : \frac{Cz_1}{y_1} : C = y_1 z_1 : x_1 z_1 : x_1 y_1$$

and the equation of the plane is

$$y_1 z_1 (x - x_1) + x_1 z_1 (y - y_1) + x_1 y_1 (z - z_1) = 0$$

which is

$$\frac{\partial V}{\partial a}(x - x_1) + \frac{\partial V}{\partial b}(y - y_1) + \frac{\partial V}{\partial c}(z - z_1) = 0$$

**An Inverse Trigonometric Identity**

**187.** [January 1954] *Proposed by B. K. Gold's Calculus Class, Los Angeles City College.*

A student differentiated $f(x) = \arcsin (x^2 - a^2)/(x^2 + a^2)$ and $g(x) = 2 \arcsin x/\sqrt{x^2 + a^2}$ and noted that their derivatives were equal. He reasoned that the anti-derivatives of $f'(x)$ and $g'(x)$ must differ only by an additive constant. Show that this is true.

I. *Solution by C. E. Jones, Tennessee A & I State University.*

Let $\sin^{-1} A - \cos^{-1} A = B$. Taking the sine of both sides and evaluating the left hand member we have $\sin B = 2A^2 - 1$ or $B = \sin^{-1}(2A^2 - 1)$. Similarly we find $\sin^{-1} A + \cos^{-1} A = \sin^{-1}(1)$. Thus adding

$$\begin{array}{r}
\sin^{-1} A - \cos^{-1} A = \sin^{-1}(2A^2 - 1) \\
\sin^{-1} A + \cos^{-1} A = \sin^{-1}(1) \\
\hline
2\sin^{-1} A = \sin^{-1}(1) + \sin^{-1}(2A^2 - 1)
\end{array}$$

If we take $A = \dfrac{x}{\sqrt{x^2 + a^2}}$ then

$$g(x) = 2\sin^{-1} \frac{x}{\sqrt{x^2 + a^2}} = \sin^{-1}(1) + \sin^{-1}\left[2x^2/(x^2 + a^2) - 1\right]$$

$$= \sin^{-1}(1) + \sin^{-1}[(x^2 - a^2)/(x^2 + a^2)]$$

or $g(x) = \sin^{-1}(1) + f(x)$ so the two functions differ by the additive constant $\sin^{-1}(1)$.

Also solved by *Norman Anning, Alhambra, California; A. L. Epstein, Cambridge Research Center, Massachusetts; R. K. Guy, University if Malaya, Singapore; John M. Howell, Los Angeles City College. Neville C. Hunsaker's Calculus Class, Utah State Agricultural College; Barbara Ann Kastner, William Smith College, New York; E. S. Keeping, University of Alberta; M. S. Klamkin, Polytechnic Institute of Brooklyn; Joseph D. E. Konhauser, Pennsylvania State College; T. F. Mulcrone, Spring Hill College, Alabama; Dennis C. Russell, Birkbeck College, University of London; S. H. Sesskin, Hofstra College, New York (two solutions); C. W. Trigg, Los Angeles City College; Chih-yi Wang, University of Minnesota; Alan Wayne, Williamsburgh Vocational High School, Brooklyn New York; and the proposers.*

**Quadrant Trisection**

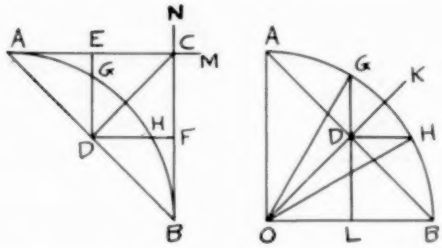
**188.** [January 1954] *Proposed by C. F. White, Naval Research Laboratory, Washington, D. C.*

Trisect a given quadrant of a circle with a draftsman's 45-45-90 triangle and T-square.

I. *Solution by C. W. Trigg, Los Angeles City College.*

To get a measure of the construction method we establish a geometrographic scale in which orienting a drawing instrument counts as one operation, and moving it into position and drawing a line counts as

another operation. The index is the total number of operations. In the following construction, the number in parentheses is the number of the operation.




1) *"Quadrant" interpreted as an arc.* Place the T-square so as to pass through the extremities $A, B$ of the arc, but not to cover the arc (1). Place the hypotenuse of the triangle against the T-sauare (2). Slide the triangle until one corner coincides with $A$, draw $AM$ (3). Slide the triangle until the other corner coincides with $B$, draw $BN$ meeting $AM$ in $C$ (4). Place leg of triangle against T-square (5). Slide triangle until leg passes through $C$, draw $CD$ (6). Place hypotenuse of triangle against T-square (7). Slide triangle until one corner coincides with $D$, draw $DE$ cutting arc $AB$ in $G$ (8). Slide triangle until other corner coincides with $D$, draw $DF$ cutting arc $AB$ in $H$ (9). Then $G$ and $H$ trisect $AB$.

2) *"Quadrant" interpreted as an area.* Place the T-square along one radius, $OB$, of the quadrant $AOB$ (1). Place the hypotenuse of the triangle against the T-square (2). Slide the triangle until one corner coincides with $B$, draw $BA$ (3). Slide the triangle until the other corner coincides with $O$, draw $OK$ cutting $BA$ in $D$ (4). Place leg of triangle against the T-square (5). Slide triangle until leg passes through $D$, draw $LD$ cutting arc $AB$ in $G$ (6). Place T-square against leg of triangle, i.e. along $LG$ (7). Slide triangle until other leg passes through $D$, draw $DH$ cutting arc $AB$ in $H$ (8). Draw $OG$ and $OH$ with aid of T-square (9) (10). Clearly, the first method could have been used here also, but drawing the trisectors $OG$ and $OH$ would have raised the operation index to 11.

*Proof:* $OK$ bisects $AB$, so $LG$ bisects $OB$. Hence $OL = \frac{1}{2}OG$, so angle $OGL$, and therefore angle $GOA$, equals 30°. By symmetry angle $HOB$ is 30° also.

It is interesting to note that with straight edge and compasses the quadrant area could have been trisected by opening the compasses to radius $OA$, drawing two arcs and two lines for an operation index of 5. To trisect the quadrant arc requires three settings of the compasses, drawing five arcs and two lines for an operation index of 10.

**II.** *Solution by Norman Anning, Alhambra, California.*

Mr. White's problem will be solved by a method which may be used for any angle. Notice that

$$x^3 - 3x - 2\cos 3A \equiv (x - 2\cos A)[x^2(2\cos A)x + (4\cos^2 A - 3)].$$

That any cubic equation and consequently the trisection equation, $x^3 - 3x - 2\cos 3A = 0$, can be solved by using two mobile right angles is well known. See Adler, *Theorie der geometrischen Konstructionen*, 1906, and pages 43, 44 of Yates, *The Trisection Problem*, 1942. To trisect any angle by using two mobile right angles:

*Construction:* Draw the circle $x^2 + y^2 = 4$. Mark the points: $A(-2,0)$, $B(2,0)$, $J(3,0)$, $C(0,2)$, $D(0,-2)$, $E(-1,0)$. At $O$ construct the angle $DOF$ which is to be trisected putting $F$ for convenience on semicircle $DBC$. Draw $FG$ parallel to $AB$ to meet the line $X = 3$ in $G$.

Make the zigzag $GHKE$ having $H$ on $AB$, $K$ on $CD$, and right angles at $H$ and $K$. This can be done by using two floating right angles.

Draw $KL$ parallel to $AB$ to meet semicircle $CAD$ in $L$. Join $LO$. Then angle $COL$ equals one-third of angle $DOF$.

*Proof:* For the proof assign coordinates $(0,p)$ to $K$, $(q,0)$ to $H$, and $(3,-r)$ to $G$ and study the zigzag $EKHG$.

The slope of $EK = p$; the slope of $KH = -p/q$; the slope of $HG = r/(q-3)$. Because $EK$ is perpendicular to $KH$ and parallel to $GH$, we have $p/1 = q/p = r/(q-3)$. From these it follows that $q = p^2$ and $r = p^3 - 3p$. If we now put $p = 2\cos A$ we have $r = 8\cos^3 A - 6\cos A = 2\cos 3A$.

The proof is complete because we have by construction linked an angle to three times that angle and, in reverse, linked an angle to one-third of that angle.

*Notes:* For the historically minded this is trisection by the Lill Process. With $EO$ as unit length the broken line $EOJG$ is the "Lill" graph of $x^3 - 3x - 2\cos 3A$. And with $EK$ as unit length the broken line $EKHG$ is

the Lill graph of $x^2 + (2\cos A)x + (4\cos A - 3)$.

The quadrant can be trisected by using just one movable right angle. When we are trisecting angle $DOB$, $G$ and $H$ fall together at $J$ and the sides of triangle $EKJ$ are the familiar 4, 2, $2\sqrt{3}$.

Also solved by *R. K. Guy, University of Malaya; John Jones Jr., Mississippi Southern College: M. S. Klamkin, Polytechnic Institute of Brooklyn; Ernest W. Peterkin, Naval Research Laboratory, Washington, D. C.; William Sanders, Mississippi Southern College; S. H. Sesskin, Hofstra College, New York: and the proposer.*

**Integration By Parts**

**189.** [January 1954] *Proposed by T. L. Miksa, Aurora, Illinois.*

Integrate

$$\int (R^2 - z^2)\ \sin^{-1}\sqrt{\frac{R^2 - b^2 - z^2}{R^2 - z^2}}\ dz$$

Solution by *Chih-yi Wang, University of Minnesota.* Let $I$ be the given integral. Integrating by parts, we get:

$$I = (R^2 z - \frac{1}{3}z^3)\ \sin^{-1}\sqrt{\frac{R^2 - b^2 - z^2}{R^2 - z^2}} + I_1$$

where

$$I_1 = \frac{b}{3}\int \frac{(3R^2z^2 - z^4)}{(R^2 - z^2)\sqrt{R^2 - b^2 - z^2}}\ dz$$

$$= -\frac{b}{3}\int \sqrt{R^2 - b^2 - z^2}\ dz - \frac{b}{3}(R^2 + b^2)\int \frac{1}{\sqrt{R^2 - b^2 - z^2}}\ dz$$

$$+ \frac{2}{3}R^4 b \int \frac{1}{(R^2 - z^2)\sqrt{R^2 - b^2 - z^2}}\ dz$$

If the substitution $z = \sqrt{R^2 - b^2}\ \sin\theta$ is made in all three integrals of $I_1$, we obtain, after simplification,

$$I = (R^2 z - \frac{1}{3}z^3)\ \sin^{-1}\sqrt{\frac{R^2 - b^2 - z^2}{R^2 - z^2}} - \frac{bz}{6}\sqrt{R^2 - b^2 - z^2}$$

$$-b\left[\frac{R^2}{2} + \frac{b^2}{6}\right]\sin^{-1}\left[\frac{z}{\sqrt{R^2 - b^2}}\right] + \frac{2}{3}R^3\ \tan^{-1}\left[\frac{bz}{R\sqrt{R^2 - b^2 - z^2}}\right] + C.$$

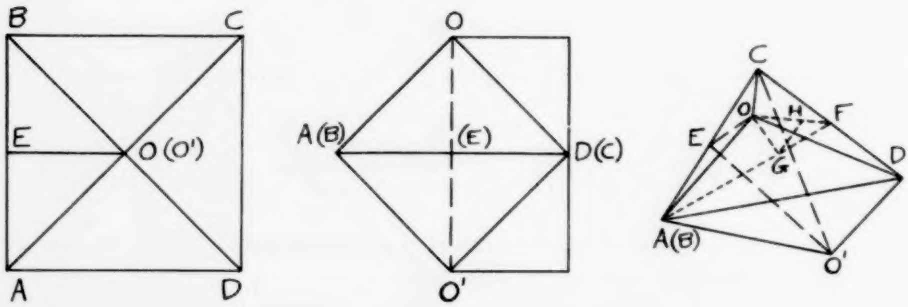
Also solved by the proposer.

## Folding a Hexahedron

**190.** [January 1954] *Proposed by C. W. Trigg, Los Angeles City College.*

(1) Show that a square envelope with edge $a$ can be folded into a stable hexahedron with congruent triangular faces. Overlapping is permitted. (2) Find the surface area of the hexahedron. (3) Find the volume of the hexahedron. (4) Find the radius of the sphere which touches all the faces.

*Solution by the proposer.*




(1) If the open edge of the envelope is $CD$, crease the envelope along its diagonals $AC$ and $BD$, which intersect at $O$ on the front, and $O'$ on the back. Crease perpendiculars $OE$ and $O'E$ from the intersections to the edge $AB$. Fold out on the semi-diagonals and in on $O'EO$ until it becomes a straight line. Bring $A$ into coincidence with $B$. Fold out along $AE$, which now falls along $BC$, and along $AD$ until the open edge $CD$ closes. The triangles $AO'O$ and $BO'O$ strengthen the faces $BOC$ and $BO'C$, thus stabilizing the hexahedron when the open edge $CD$ is sealed.

(2) A typical face of the hexahedron is triangle $BOA$. Hence the surface area is $6[a(a/2)/2]$ or $3a^2/2$.

(3) The hexahedron consists of two triangular pyramids, $O$-$ACD$ and $O'$-$ACD$ which have a common face — an equilateral triangle with side $a$. The exposed faces are isosceles right triangles with legs $a/\sqrt{2}$. Since the three face angles of the trihedral angle $O$-$ACD$ are right angles, the volume of the hexahedron is:

$$2\left[\frac{1}{3}\cdot\frac{1}{2}(a/\sqrt{2})^2(a/\sqrt{2})\right] \quad \text{or} \quad a^3\sqrt{2}/12.$$

(4) The plane through $AO$ and $AO'$ bisects $DC$ in $F$. In the triangle $AOF$, the foot of the altitude $OG$ is the centroid of equilateral triangle $ACD$. Hence $GF = (a\sqrt{3}/2)/3$ and $OF = a/2$, so $OG = a/\sqrt{6}$. Now the perpendicular $GH$ from $G$ to $OF$ is the radius of the inscribed sphere. Since $GH/GF = OG/OF$, then $GH = (a/2\sqrt{3})(a/\sqrt{6})/(a/2)$ or $a\sqrt{2}/6$.

### A Binomial Sum

**191.** [January 1954] *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Find the sum

$$\sum_{s=0}^{n} (-1)^s \left[ \frac{1}{s+1} + \frac{1}{s+2} + \cdots + \frac{1}{s+n} \right] \binom{n}{s}$$

**I.** *Solution by L. Carlitz, Duke University.*

Put

$$S_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k} \quad (k \geq 1), \qquad S_0 = 0.$$

Then

$$S = \sum_{s=0}^{n} (-1)^s \left[ \frac{1}{s+1} + \frac{1}{s+2} + \cdots + \frac{1}{s+n} \right] \binom{n}{s} = \sum_{s=0}^{n} (-1)^s (S_{n+s} - S_s) \binom{n}{s}$$

Using the notation of finite differences

$$\Delta^n S_k = \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} S_{k+s}$$

But

$$\ddot{\Delta} S_k = \frac{1}{k+1}, \quad \Delta^2 S_k = \frac{-1}{(k+1)(k+2)}, \quad \cdots$$

$$\Delta^n S_k = (-1)^{n-1} \frac{(n-1)!}{(k+1)(k+2)\cdots(k+n)},$$

so that

$$\sum_{s=0}^{n} (-1)^{n-s} (S_{k+s} - S_s) \binom{n}{s} = (-1)^{n-1} \frac{(n-1)!}{(k+1)\cdots(k+n)} - \frac{(n-1)!}{n!}$$

In particular for $k = n$ we get

$$S = (-1)^n \sum_{s=0}^{-n} (-1)^{n-s} (S_{n+s} - S_s) \binom{n}{s}$$

$$= (-1)^{-1} \left\{ \frac{(n-1)!}{(n+1)\cdots(2n)} - \frac{(n-1)!}{n!} \right\} = \frac{1}{n} \left\{ 1 - \binom{2n}{n}^{-1} \right\}$$

**II.** *Solution by Dennis C. Russell, Birkbeck College, University of London.*

Let

$$S = \sum_{s=0}^{n} (-1)^s \left[ \frac{1}{s+1} + \frac{1}{s+2} + \cdots + \frac{1}{s+n} \right] \binom{n}{s}$$

$$= \sum_{s=0}^{n} \sum_{r=1}^{n} \frac{(-1)\binom{n}{s}}{s+r} = \sum_{r=1}^{n} \sum_{s=0}^{n} \frac{(-1)\binom{n}{s}}{s+r} \tag{1}$$

Write

$$f_r(x) = \sum_{s=0}^{n} \frac{(-1)^s \binom{n}{s} x^{s+r}}{s+r},$$

so that

$$f_r'(x) = \sum_{s=0}^{n} (-1)^s \binom{n}{s} x^{s+r-1} = x^{r-1} \sum_{s=0}^{n} \binom{n}{s} (-x)^s = x^{r-1}(1-x)^n$$

Hence

$$f_r(x) = \int_0^x t^{r-1} (1-t)^n \, dt \tag{2}$$

thus from (1) and (2),

$$S = \sum_{r=1}^{n} f_r(1) = \sum_{r=1}^{n} \int_0^1 t^{r-1} (1-t)^n dt = \int_0^1 (1-t)^n \sum_{r\ 1}^{n} t^{r-1} dt$$

$$= \int_0^1 (1-t)^n \left[\frac{1-t^n}{1-t}\right] dt = \int_0^1 (1-t)^{n-1} dt - \int_0^1 t^n (1-t)^{n-1} dt$$

$$= \frac{1}{n} - \frac{n!(n-1)!}{(2n)!};$$

using

$$\int_0^1 t^p (1-t)^2 dt = \frac{p!\,2!}{(p+q+1)!}, \quad \text{i.e.} \quad S = \frac{1}{n}\left\{1 - \frac{(n!)^2}{(2n)!}\right\}.$$

Also solved by *Huseyin Demir, Zonguldak, Turkey; H. W. Gould, Portsmouth, Virginia; Richard K. Guy University of Malaya, Singapore; Chih-yi Wang, University of Minnesota; and the proposer.*

### Triangle Symmetries

**192.** [January 1954] *Proposed by V. Thebault, Tennie, Sarthe, France.*

If $A'$, $B'$, $C'$ are the symmetries of the vertices of a triangle $ABC$ with respect to a fixed point, the circumcircles of the three triangles $AB'C'$, $BC'A'$, $CA'B'$ have a point in common which lies on the circumcircle of the triangle $ABC$.

*Solution by Huseyin Demir, Zonguldak, Turkey.* It will suffice to prove that any two of the circumcircles intersect on the circumcircle $(O)$ of $ABC$. Let $I$ be the intersection of the circles $BC'A'$ and $CA'B'$. To prove that it belongs to $(O)$ we show that $\angle BIC = \angle BAC = \angle A$: $\angle BIC = \angle BIA' + \angle A'IC = \angle BC'A' + \angle A'B'C = \angle B'CA + \angle KB'C = \angle B'CK + \angle KB'C = \angle A'KC = \angle BAC = \angle A$.

The first equality follows from the fact that the points $B, I, C', A'$ on

one hand and $B, B', C, A'$ on the other lie on the respective circumcircles, and the other equalities from the parallelisms:

$$BC' \parallel CB', A'C' \parallel AC, A'B' \parallel AB.$$

Also solved by *H. E. Fettis, Dayton, Ohio; O. J. Ramler, Catholic University of America and the proposer.*

## QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

**Q 117** How many squares are there on a chessboard? [*Submitted by Huseyin Demir*]

**Q 118** Find the three smallest integers such that the sum of the reciprocals of its divisors equals 2. [*Submitted by M. S. Klamkin*]

**Q 119** Prove that $(21/20)^{100} > 100$. (No tables please!) [*Submitted by Chih-yi Wang*]

**Q 120** We may write $n! = 2^k(2M-1)$. Show that $k < n$. [*Submitted by Richard C. Bartell*]

**Q 121** If $\left|z^n + a_1 z^{n-1} + \cdots + a_n + \dfrac{b_1}{z} + \dfrac{b_2}{z^2} + \cdots + \dfrac{b_r}{z^r}\right| = 1$ for $|z| = 1$, what are the restrictions on the coefficients $a_i$ and $b_i$? [*Submitted by M. S. Klamkin*]

## ANSWERS

$$k = \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \left[\frac{n}{2^3}\right] + \cdots + \left[\frac{n}{2^p}\right] \le \frac{n}{2} + \frac{n}{2^2} + \cdots + \frac{n}{2^p} <$$

**A 117.** Not 64, but $1^2 + 2^2 + \cdots + 8^2 + \dfrac{8 \cdot 9 \cdot 17}{6} = 204$.

**A 118.** $\sum 1/d_r = 2$. But $\sum N/d_r = \sum d_r$. Thus, $\sum d_r = 2N$ or $N$ is a perfect number and the answer is 6, 28 and 496.

**A 119.** It suffices to show $(21/20)^{50} > 10$. Write $(21/20)^{50} = (1 + 1/20)^{50}$. Expand by the Binomial Theorem. Since all terms are positive the first five terms are sufficient as:

$$(1 + 1/20)^{50} = 1 + 2.5 + 3 + 2.4 + 1.4 > 10.$$

**A 120.** Let $[x]$ represent the greatest integer less than or equal to $x$. Then we have

$$n\left\{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^p}\right\}=n \quad \text{where} \quad 2^p \le n < 2^{p+1}$$

**A 121.** The expression is equivalent to:

$$|b_r z^{r+n} + b_{r-1} z^{r+n-1} + \cdots + a_1 z + 1| = 1 \quad \text{for} \quad |z| = 1.$$

By the Maximum Modulus Theorem all the coefficients must be zero.

## Quickening The Quickies

From time to time readers submit answers to the Quickies which are shorter, neater or more novel than those published. Such answers are welcome and will appear in subsequent issues.

**Q 101** [November 1953] Let $c=u^2$ and $d=v^2$. Then $(c+d)^3 \equiv c(c+d)^2+d(c+d)^2$. This gives $z=c+d$ and $x=u(c+d)$, $y=v(c+d)$. [*Submitted by A. L. Epstein.*]

**Q 104** [January 1954] The given numbers can not be prime pairs as 1,000,000,011 is divisible by 3 (the sum of its digits is 3) and it therefore is composite. [*Submitted separately by H. M. Gehman, Richard K. Guy and C. W. Trigg*]

**Q 105** [January 1954]

$$\Sigma[n^2-(2p-1)n] = \Sigma\,[n^2-2pn+n] = (n^2+n)\,\Sigma\,1-2n\,\Sigma\,p$$

$$= (n^2+n)\,n-2n\left[\frac{n(n+1)}{2}\right] \equiv 0.$$

This proof depends only upon the formula for the sum of an Arithmetic Progression. [ *Submitted separately by H. M. Gehman and Richard K. Guy.*]

**Q 105** [March 1954] The following geometrical solution makes no appeal to other results.

| | | | |
|---|---|---|---|
| *abbbbb* | *abbbbb* | *abbbbb* | *abbbbb* |
| *aabbbb* | *aabbbb* | *aabbbb* | *accccc* |
| *aaabbb* | *aaabbb* | *aacccc* | *accccc* |
| *aaaabb* | *aaaccc* | *aacccc* | *accccc* |
| *aaaacc* | *aaaccc* | *aacccc* | *accccc* |

The four arrays represent $n=4$ layers of $(n+1)(n-2)$ cubes. They include three pyramids $a$, $b$ and $c$ each of $1\cdot 2+2\cdot 3+\cdots+n(n+1)$ cubes. So $\Sigma\, n(n+1) = (1/3)n(n+1)(n-2)$. [*Submitted by Richard K. Guy*]

# MISCELLANEOUS NOTES

*Edited by*

Charles K. Robbins

**Articles intended for this Department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.**

**Editor, "Mathematics Magazine"**

Dear Sir:

I do not know whether or not you are content in having the present circulation of your publication remain more or less static and of interest to the present group of readers whose mathematical maturity renders the articles comprehensible to them, or whether you would not be averse to a wider interest and greater circulation.

There is nothing reprehensible in a limited circulation maintained for a select group. It may, in fact, be highly desireable, yet I believe that it is possible to develop interest in mathematics in a large segment of the population which your publication does not now reach because, for the most part, the articles contained therein are on a plane inacessible to such readers. I *know* I am such an individual.

I have subscribed now for two years (and intend to continue) though I do not know just *why*, as it is not often I see an article which I can grasp. In my own case I have gone through Differential and Integral Calculus (and there is a goodly number of individuals extant with this background) and have a strong liking for mathematics as a recreation. I have been interested in exploring mathematical fields beyond my formal training simply for the pure pleasure I derive therefrom. As a consequence I have "loaded up" on a goodly number of texts and treatises and endeavored to "whack away" (unassisted) at them in the endeavor to obtain some insight into mathematics. It is indeed a rare text whose author has the gift (which is no doubt divinely conferred since it is so rare) to *very clearly*, step by step, and ***WITHOUT IMPORTANT OMISSIONS*** explain new concepts in a comprehensible manner to one unfamiliar with them.

Now it appears to me that if you carried in "Mathematics Magazine" a *single article* in each issue of great clarity and simplicity on any of a number of subjects there would soon be apparent a decided increase in interest in your publication and a concomitantly wider circulation.

For example, there are the fields of Cantor's Infinities, Symbolic Logic, Vectors, Infinite Series, Congruences and what not. When I had finished my college training I had only the grogiest idea that these existed. The little more I *now* know of them, however, does not leave me unaware of the vast extent these fields cover. I have however learned

that it *is* possible to obtain *some* idea of what they are about and that this *can* be achieved ***IF THE EXPLANATION IS SUFFICIENTLY DETAILED.*** Such detail is absent in most texts which are, after all, intended to be supplemented by class instruction. It appears to me that if each of your issues would have one article on an elementary phase of one of the multitude of fascinating but little known (except among the professionals) fields of mathematics the response in the matter of enhanced interest would be immediate.

I still entertain the hope that there are a sufficient number of others like myself who enjoy mathematics for its own sake and are eager to pursue it further; who rejoice to the *n*th degree when they come across something new in mathematics which is put within their reach, and I suggest Mathematics Magazine as a media.

Like a Kid looking in a candy store I see a multitude of delectable things in Mathematics Magazine but have no way of "getting at them" as the greater part of them are beyond me. How about an occasional article for us eager (though pigmie) beavers which we can gnaw at with our less developed teeth?

Sincerely yours,

E. L. Dimmick, M. D.

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Dear Dr. Dimmick:

The following extract from a letter which we are just mailing to prospective subscribers contains my feeling about the matter brought up in your letter. Our trouble will be to find people who can and will translate mathematics into language which can be read, generally.

Your suggestion* for a discussion of arithmetic and geometric series "in particular the formation of the general terms" is good, other topics that might be of general interest are complex numbers, continued fractions, algebraic identities, equations in one variable, derivatives etc., or whatever a prospective contributor is interested in discussing.

'In every issue of the Mathematics Magazine you will find both popular (a new feature) and Technical Articles, Problems and Questions with fascinating "Quickies" and "Trickies", Miscellaneous Notes, Biographical Sketches of our authors et cetera.

One subscriber writes "I have liked your magazine, its informal, friendly tone. I have used it in classes on several levels to stimulate reading for pleasure by intelligent people not specifically interested in the technical mathematics."

A couple of centuries ago "The Queen of the Sciences" was read quite generally by thoughtful laymen. Now it is relegated to the specialist, largely, I believe, because of its highly condensed language. Consequently it is being discarded as a required course in high school. This is suicide for mathematics in the long run and is a great loss to the general public, in whose daily life mathematics is second to no Science in importance.

*This suggestion received in a second letter from Dr. Dimmick.

The Mathematics Magazine is doing considerable toward correcting this dangerous development, if we can judge from our correspondence. We invite you to join in this undertaking by subscribing and perhaps contributing on some of the topics suggested in this September-October issue.'

Sincerely yours,

Glenn James

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## CONGRUENCE*

The study of Congruence besides forming a major topic in Number Theory and other fields of pure mathematics finds practical application in such problems as arise in calculation of calendars and calendar dates, the checking of plain arithemetical multiplication by the "casting out of nines", in determining whether large numbers are divisible by other numbers (e.g. is $2^{32}$ divisible by 641?) and other manifold applications.

It is the purpose of this article to present a clear, elementary understanding of the basis of congruence.

. . . . . . . . . . . . .

Congruence deals fundamentally with division of one number by an other and the relations arising out of such divisions.

Let $m$ be any positive integer (i.e. any positive "whole number").

Let this number, $m$, be called the ***MODULUS***.

Let us choose any other two integers $a$ and $b$, which may be either positive or negative.

Suppose we subtract $b$ from $a$ and find that after the subtraction the result can be divided evenly by the number $m$ (the Modulus), thus

$$\frac{a-b}{m} = \text{an integer (a whole number)}$$

then $a$ and $b$ are said to be "congruent modulo $m$" and this is expressed mathematically by $a \equiv b \pmod{m}$ or by $a - b \equiv 0 \pmod{m}$.

Example 1. $20 \equiv 5 \pmod 3$. Twenty is congruent to five modulo three since $20 - 5 = 15$ and 15 is divisible evenly by 3.

Example 2. $7^2 \equiv -1 \pmod 5$. $7^2 = 49$ so 49 is congruent to minus one modulo five. For $49 - (-1) = 49 + 1 = 50$. Since 50 is divisible by 5, $7^2 \equiv -1$ (mod 5).

It is thus seen that congruence is a relationship established between two numbers by virtue of the fact that the difference between these two numbers is divisible by a third number called the MODULUS.

*Invited by the editor to illustrate what Dr Dimmick means in his letter above.

Suppose we let $a$ equal any integer and $m$ (as before) any modulus. If $a$ is divided by $m$ there will, after the division has been performed, be a "remainder". If the division is exact (i.e. if it "comes out even") the remainder will be 0. If it does not "come out even" the remainder will be a number 1, 2, 3, 4, $\cdots$ or some positive integer WHICH IS ALWAYS LESS THAN $m$ (the divisor). Try dividing any whole number by three, for example, and the remainder will always be *less than* three. In this case the remainder will always be one of the three numbers 0, 1, or 2.

If we write in arithmetic form the statement "$a$ divided by $m$ gives a quotient $q$ and a remainder $r$ it will assume the form:

$$\begin{array}{r} \underline{|q} \\ m)\ a \\ \underline{mq} \\ r \end{array}$$

In Algebraic form this is written $a/m = q + r/m$ or $a = mq + r$

In either case $a$ is any integer, $m$ is any other integer, $q$ is the quotient obtained when $a$ is divided by $m$, $r$ is the remainder.

Example:

$$\begin{array}{r} \underline{|2} \\ 4)\ 9 \\ \underline{8} \\ 1 \end{array} \qquad 9/4 = 2 + 1/4 \quad \text{or} \quad 9 = 4 \times 2 + 1$$

Such a division of $a$ by $m$ leaving a remainder $r$ sets up a congruence $a \equiv r \pmod{m}$ and in this congruence $r$ *is always less than* $m$* (expressed algebraically $r < m$). In this relation $r$ is called "THE LEAST POSITIVE *RESIDUE* OF $a$ mod $m$".

This congruence relationship, $a \equiv r \pmod{m}$ forms the basis for *separating numbers into "groups" or "classes"* distinguished from one another by distinct mathematical properties illustrated as follows:

Suppose we select a number arbitrarily to be the modulus – let us say the number 3. Let us begin dividing successive integers by this modulus:

| | |
|---|---|
| 3 ÷ 3 = 1 + 0 remainder | 7 ÷ 3 = 2 + 1 remainder |
| 4 ÷ 3 = 1 + 1 " | 8 ÷ 3 = 2 + 2 " |
| 5 ÷ 3 = 1 + 2 " | 9 ÷ 3 = 3 + 0 " |
| 6 ÷ 3 = 2 + 0 " | etc. |

It will be noted that:

the numbers 3, 6 and 9 when divided by 3 give 0 for remainder
the numbers 4, 7 and 10 when divided by 3 give 1 for remainder
the numbers 5, 8 and 11 when divided by 3 give 2 for remainder

Thus if the modulus is chosen as 3 there are *three* classes of numbers each class of which gives like remainders on being divided by 3:

Class I – All numbers giving a remainder of 0 when divided by 3.
Class II – All numbers giving a remainder of 1 when divided by 3.
Class III – All numbers giving a remainder of 2 when divided by 3.

If we had chosen the number 4 to be the modulus and divided successive integers by 4 it would have been found that there were FOUR Classes of remainders 0, 1, 2, and 3. The number of classes which result from dividing by any integer will always be *the same* as the integer by which we divide. These classes are called "RESIDUE CLASSES". In the above example where we chose 3 as a modulus (i.e. as a divisor) we say, "There are THREE RESIDUE CLASSES (mod 3) ". In general THERE ARE $m$ RESIDUE CLASSES (mod $m$). For a given remainder, $r$, the Residue Class to which it belongs consists of the numbers $r$, $r+m$, $r+2m$, $r+3m$, $\cdots$, etc.

It is possible (and often more convenient) to express the congruence $a \equiv b \pmod{m}$ in a slightly different manner by saying "$a$ is congruent to $b$ when it differs from $b$ by a multiple of $m$", i.e. $a = b + km$. In common language this equation states that in order to equal $a$ you must add to $b$ either $m$, $2m$, $3m$ or some whole number $k$ times $m$.

The two notations $a \equiv b \pmod{m}$ and $a = b + km$ thus express the same relationship between $a$ and $b$.

If $a \equiv r \pmod{m}$, i.e. $a = km + r$, where $r$ is the remainder in the formation of residue classes (mod $m$) and $a \equiv b \pmod{m}$ i.e. $a = k'm + b$, then $b - r + m(k' - k) = 0$ or $b \equiv r \pmod{m}$. Hence we may say that if $a \equiv b$ (mod $m$) they belong to the same *residue class* (mod $m$).

The following notations are a summation and extension of what has thus far been discussed:

1. $a \equiv b \pmod{m}$ means $a - b$ is divisible by $m$.
2. $a \not\equiv b \pmod{m}$ means $a$ is NOT congruent to $b$ (mod $m$), i.e. $a - b$ is not divisible by $m$.
3. $a \equiv r \pmod{m}$ means $r$ is the remainder obtained when $a$ is divided by $m$.
4. $a \equiv 0 \pmod{m}$ means that $a$ is exactly divisible by $m$.
5. Any EVEN number $n$ is expressed in Congruent notation by $n \equiv 0$ (mod 2).
6. Any ODD number $w$ is expressed in Congruent notation by $w \equiv 1$ (mod 2).

E. L. Dimmick, M. D.

# THE NUMBER SYSTEM OF THE MAYAS

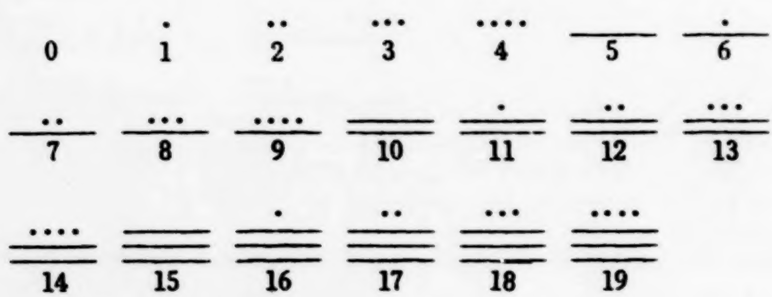
Gary D. Salyers

In the year 1517 when Francisco de Cordaba landed the first Spanish expedition on the coast of Yucatan in southern Mexico he found evidence of a great Indian civilization that had received its deathblow in wars several centuries before. In this, Mayan civilization had developed a very good number system and some brilliant discoveries by these Indians whose cycle, some investigators have asserted, began as early as 3373 B. C.[1]

The Maya number system was the best that was developed in the Americas, and was based on the calendar. The Mayas were great calendar makers and their "calendar script, which is the only number language based on the same principle as the Hindu notation, also uses horizontal bars placed one above another."[2]

The Mayan calendar numerals used seperate signs for the numbers one through five, for twenty, and for four hundred (the same symbol was used sometimes for 360 and sometimes for 400).[3] Some explanation of this 360-400 deal is given by Morley who states, "the break in the third order of units ··· 360 instead of 400, the latter of which is the correct value of the third term in a strictly vigesimal (based on twenty) system, however, was used only in counting time; in counting everything else the Maya followed the vigesimal progression consistently—1, 20, 400 (instead of 360), 8,000 (instead of 7200), 160,000 (instead of 144,000), 3,200,000 (instead of 2,880,000), and so on."[4]

The first twenty numbers of the Maya and their Hindu-Arabic notation are as follows:




1. David Smith, *History of Mathematics*, p. 43.
2. Lancelot Hogben, *Mathematics for the Million*, p. 196.
3. *Ibid.*, p. 41.
4. Sylvanus G. Morley, *The Ancient Maya*, p. 276.

To write numbers above 19 or above the first order of units the Mayas used a positional systém of numeration with the values increasing from bottom to top. These positional values increased by twenties except for the third, which in the counting of time alone, is only 18 instead of 20 times the second position. For example consider the number 20. This is 1 complete unit of the second order and no units of the first order, and so needs two symbols, a symbol for zero in the first or lowest position to show that no units are involved in the number, and one unit of the second order written

•

•

The number 37 was written

•

• •

———
———
——— ,

this is 17 units of the first order and 1 unit of the second order (17 + 20). The number 360 is the third order of units and is written

•

(0 + 0 + 360). The number 7,202 is written

•

• •

this is 2 units of the first order, 0 units of the second order, 0 units of the third order, and 1 unit of the fourth order $(2 + 0 \cdot 20 + 0 \cdot 360 + 1 \cdot 7{,}200)$.[5]

Karpinski,[6] in writing about the position in the Maya number system, says, "to make the system easily applicable to computation it is essential that ··· the nineteen in a twenty system, should be independent like the letter symbols and not compounded. Thus ═══ in Maya represents 10; not 105 (5 twenties and five) as they would in a pure place system. Each of these symbols occupies two places instead of one."

According to Hogben[7] the discovery by the Mayas of the zero as a placeholder took place about A.D. 500 and this was made independently of the Hindus who are generally credited with the discovery. Morely gives the Maya more credit as he states. "the Maya developed man's first positional arithmetical system, one involving the concept of zero;

5 *Ibid.*, pp. 280-281.
6 Louis Karpinski, *The History of Arithmetic*, pp. 40-41.
7 Hogben, *op. cit.*, p. 288.

this is among the most brilliant intellectual achievements of all time."[8]

The Maya calendar on which the number system was based had names for each of the individual parts. "The unit of the Maya calendar was the day or kin. The second order of units, consisting of 20 kins, was called the uinal. In a perfect vigesimal system of numeration, the third term should be 400, that is 20 · 20 · 1, but at this point, in counting time only, the Maya introduced a single variation, in order to make the period of their order agree in length with their calendar year as nearly as possible. The third order of the Maya vigesimal system, the tun, therefore, was composed of 18 (instead of 20) uinals, or 360 (instead of 400) kins; 360 days or kins being a much closer approximation to the length of the 365-day calendar year than 400 days."[9]

Above the kins or third order, however, "the unit of progression used to form all the higher numbers is uniformly 20, as will be seen below, where the names and numerical values of the nine known different orders of time periods are given:

| | |
|---|---|
| 20 kins | = 1 uinal or 20 days |
| 18 uinals | = 1 tun or 360 days |
| 20 tuns | = 1 katun or 7,200 days |
| 20 katuns | = 1 baktun or 144,000 days |
| 20 baktuns | = 1 pictun or 2,880,000 days |
| 20 pictuns | = 1 calabtun or 57,600,000 days |
| 20 calabtuns | = 1 kinchiltun or 1,152,000,000 days |
| 20 kinchiltuns | = 1 alantun or 23,040,000,000 days."[10] |

The Maya had a sacred year of 260 days, an official year of 360 days, a solar year of 365+days ··· the lowest common multiple of 260 and 365, or 18,980, was taken by the Maya as the "calendar round," a period or 52 years, which was important in Maya chronology.[11]

The Maya like the Egyptians used stone objects made to "show how the sun's position at sunrise or sunset on the solstices was fixed by alignment with two pillars of different height."[12] The Mayas also made many buildings, temples, and ball courts which required considerable knowledge of mathematics.

The Mayas also used picture writing for numbers. They had a symbol of a special head to represent each of the first thirteen numbers and zero, which were regarded as primary numbers. The head-variant for 10 is the death's head, or skull and from 14 to 19 the fleshless lower jaw of the death's head was the part used to represent the value of 10.

8 Morley, *op. cit.*, p. 454.
9 *Ibid.*, p. 275.
10 *Ibid.*, p. 276.
11 Florian Cajori, *A History of Mathematics*, p. 70.
12 Hogben, *op. cit.*, p. 56.

Thus if the fleshless lower jaw was applied to the lower part of the head which represented 6 the resulting head would be that for 16.[13] Morley says (p. 280) that it is not improbable that the 13 heads represented the 13 numbers from 1 to 13 inclusive and probably those of the Oxlahuntiku or Thirteen Gods of the Upper World as opposed to the Bolontiku or Nine Gods of the Lower World. Each of the former was associated with one of these thirteen numbers, being its especial patron.

In picture writing twenty of anything, for example a basket, was a picture of a basket with a flag flying from it. Other similar types of symbols with other things like spikes and others, meant that there were so many of each drawing in one symbol. An example of this would be five jars of honey with a flag flying from each. This would mean that each jar of honey with the flag were 20 jars of honey, so the five jars added together would be 100 jars of honey. This is similar to our tally system.[14]

The Maya like many other primitive peoples believed that there was something magic about numbers and from ancient times believed in lucky and unlucky numbers. "Nine is, and always has been, especially lucky, perhaps because of its association with ancient Maya heaven··· Thirteen is another lucky Mayan number, perhaps because of its important function in the ancient Maya calendar···· The use of thirteen as a lucky number, however, seems to be confined chiefly to religious ceremonies···"[15]

In ancient times the Mayas used limestone to carve their numbers on, but "the fact that toward the end of the ninth century A.D. the Maya no longer erected their characteristic carved and dated *stalae* suggests that by then they had largely turned to the more readily handled media of paper and books for recording ··· data."[16] The Maya paper was called huun-paper and was far superior in both texture and durability to Egyptian papyrus.[17]

Smith states that "there is no evidence in any extant record that it (the Maya number system) was used for purposes of computation, its use in texts being merely to express the time elapsing between dates."[18] Reves, however, doesn't agree with Smith and says that the Maya had "developed the writing of large numbers (the highest number found in our decimal notation 12,489,781[19]); no fractions, but did long numeral computations of multiplication and division; and applications to the calendar." [20]

13 Morley, *op. cit.*, p. 279.
14 Karpinski, *op. cit.*, p. 32.
15 Morley, *op. cit.*, pp. 36-37
16 Slusser, *op. cit.*, p. 36
17 *Ibid.*
18 Smith, *op. cit.*, p. 45.
19 Cajori, *op. cit.*, p. 69.
20 George E. Reves, "Outline of the History of Arithmetic," *School Science and Mathematics*, p. 611.

Smith also suggests that the reason there wasn't any computation was because the Maya had the abacus which he says may have been brought there by the first people who came from Asia.[21] Whether this is true or not will probably always remain a mystery.

21 Smith, *op. cit.*, p. 45.

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# ROUND TABLE ON FERMAT'S LAST THEOREM

Dear Mr. James:

Following your suggestion in Mathematics Magazine Vol. 27, No. 4, March-April, 1954, we submit to you the enclosed very short and elementary paper refering to an approach to the Last Theorem of Fermat the probability way.

Sincerely yours,

Dr. P. H. Heimann
Dr. Fred G. Elston

The Last Theorem of Fermat is not only a pure problem of algebraic analysis but also a probability problem.

In the series of 1 to $n^r$ are contained $n$ consecutive integers of $r$th power. If we combine these integers in sums of 2 there are $n(n-1)/2$ possible combinations. The result of each of such sums must not be necessarily contained in the series from 1 to $n^r$ but is certainly contained in the series from 1 to $2n^r$.

What is now the probability to take out by random choice a definite number of the $r$th powers of the series of 1 to $2n^r$? This probability calculated as the quotient of all favorable cases divided by all possible cases is

$$p = \frac{\sqrt[r]{2}\, n}{2n^r} = \frac{1}{2^{(r-1)/r}\, n^{r-1}}.$$

If the sums $x^r + y^r$ contained in the series from 1 to $2n^r$ can be considered as a random choice then the possible number of $r$th powers from these sums equals the number of combinations times the probability or, letting $w$ = the probable number of $r$th powers,

$$W = \frac{n(n-1)}{2\; 2^{(r-1)/r}\, n^{r-1}} = \frac{n-1}{2^{(2r-1)/r}\, n^{r-2}}.$$

This equation shows that for any value of $r > 2$ the probable number of sums

$$x^r + y^r = z^r \quad (x, y, z \text{ integers})$$

is a prime fraction smaller than 1.

Example: Let $r = 3$. Then

$$W_3 = \frac{n-1}{2^{5/3}\, n}$$

If $n$ is large,

$$W_3 \doteq \frac{1}{2^{5/3}}$$

It further follows that with increasing $r$ the fractions decrease in value because the denominator contains $n$ or higher powers of $n$.

The special case $r = 2$ leads to an interesting conclusion for the number of Pythagorean triples in the series from 1 to $2n^2$:

$$W_2 = \frac{n-1}{2^{3/2}\, n^0} = \frac{n-1}{\sqrt{8}}\ !$$

Dr. P. Heinrich Heimann
Dr. Fred G. Elston

Ed's note: This paper arrived just in time to be squeezed into the first issue of Vol. 28, No. 1. We are publishing it without refereeing and are inviting our readers to discuss it in future issues.

# CURRENT PAPERS AND BOOKS

*Edited by*

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

*Concerning Sin* $(A+B)$

(F. H. Young - Vol. 27, No. 4)

Dear Mr. James:

I was very interested in F. H. Young's method for deriving the formulae for sin $(A+B)$ etc., in the March-April, 1954 Mathematics Magazine. There is one snag which I should very much like to have cleared up:

$$\text{“}\sin (A+B) = \pm (\sin A \cos B + \cos A \sin B).$$

*Since this is to be an identity* for all $a$ and $b$ $\cdots$"

Assuming the order of treatment suggested, I can't see why this should be an identity - might it not be + sometimes and - others? (cf. sin $A/2 = \pm\sqrt{(1-\cos A)/2}$ .)

I expect the author has a simple answer to this, but if it were a question of dragging in sign-conventions, his method would be no quicker than the traditional one.

I very much hope there is a short answer to this, as his method is so good.

Yours truly,

G. Matthews
St. Dunstan's College
Catford, England

Dear Mr. Matthews:

Suppose that both signs were needed. Then for some $A$ and $B$, sin $(A+B) = -(\sin A \cos B + \cos A \sin B)$ with $A+B \not\equiv 0 \pmod{\pi}$. From the continuity of $\sin x$, either $A$ or $B$, say $A$, can be changed to $\equiv 0 \pmod{\pi}$ without changing the sign of sin $(A+B)$. Since the + sign is obviously required when $A \equiv 0 \pmod{\pi}$ for all $B$, a contradiction is reached. Recall that the sine and cosine functions are assumed defined as coordinates on the unit circle. This continuity argument was implied but

perhaps too briefly or too eingmatically stated in the paper as published.

Sincerely yours,
F. H. Young
Montana State College

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*Concerning The Parabola of Surety*

(R. F. Graesser - Vol. 27, No. 4)

Your correspondent R. F. Graesser, in his article on "The Parabola of Surety" in your March-April, 1954 issue of the Mathematics Magazine, is guilty of some rather muddled thinking, or at least loose terminology. The conclusion at which he arrives is of course true, and the actual processes, shorn of the explanations, valid; but he has not avoided Scylla and Charybdis, for if one examines these explanations they are easily seen to be misleading. The major error is in the statement that the parabola of surety is the same as the locus of the *highest* point attainable by the gun, whereas it is in fact the locus of the *furthest* points attainable. Your correspondent purports to prove the former, which is not in fact a parabola at all, as is seen by the following calculation (I write $\alpha$ as the angle of inclination of the gun, in place of $\theta$, as I wish to use $\theta$ later).

The equation of the trajectory is

$$y = x \tan\alpha - \frac{g x^2 \sec^2\alpha}{2v_0^2}$$

By equating $\frac{dy}{dx}$ to 0 in the usual way, to obtain the highest point on this trajectory, we readily obtain

$$x = \frac{v_0^2}{g}\sin\alpha\cos\alpha \qquad y = \frac{v_0^2}{2g}\sin^2\alpha$$

Writing $l = v_0^2/g$, elimination of $\alpha$ then gives the locus of the *highest* points attainable by the gun to be the ellipse

$$(x/l)^2 + [(2y - l)/l]^2 = 1.$$

I do not think it possible to give an elementary explanation of the process used in the article you published (viz. partial differentiation with respect to a parameter). If an elementary approach is needed, it can only be from the fact that the enveloping parabola (or Parabola of Surety) is the locus of the *furthest* points attainable by the gun, and we may do this by the following method which does not, in fact, even use the calculus:

The equation of the trajectory is

$$y = x \tan\alpha - \frac{g x^2 \sec^2\alpha}{2v_0^2} \qquad \text{i}$$

To obtain the range $R$ in a direction making an angle $\theta$ with the vertical, we let $(X,Y)$ be the coordinates of the end of this range, so that $X,Y$ satisfy i, and $Y = X\cot\theta$. Hence

$$Y = X\cot\theta = X\tan\alpha - \frac{g x^2 \sec^2\alpha}{2v_0^2},$$

$$X = \frac{2v_0^2}{g}\cos^2\alpha\,(\tan\alpha - \cot\theta).$$

Then

$$R = X\operatorname{cosec}\theta = \frac{2v_0^2}{g}\,\frac{\cos\alpha}{\sin^2\theta}\,(\sin\alpha\sin\theta - \cos\alpha\cos\theta)$$

$$= -\frac{2v_0^2}{g}\operatorname{cosec}^2\theta\,\cos\alpha\cos(\alpha+\theta)$$

$$= \frac{v_0^2}{g}\operatorname{cosec}^2\theta\,(-\cos(2\alpha+\theta) - \cos\theta)$$

As $\alpha$ varies, this expression is a maximum when

$$\cos(2\alpha+\theta) = -1 \quad \text{or} \quad 2\alpha + \theta = \pi;$$

then substituting in ii this makes the *maximum range* $r$ in the direction $\theta$ to be

$$r = \frac{v_0^2}{g}\operatorname{cosec}^2\theta(1 - \cos\theta).$$

Writing $l = v_0^2/g$, this is easily simplified to the form

$$\frac{l}{r} = 1 + \cos\theta \qquad \text{iii}$$

This equation iii is the polar equation of a parabola with focus at the origin, axis the y axis and semi-latus-rectum $l$. If required, we can readily deduce the Cartesian equation of the enveloping parabola from $x = r\sin\theta$, $y - r\cos\theta$ (remembering that $\theta$ is measured from the vertical), for iii then gives

$$l = r + r\cos\theta = r + y \quad \text{or} \quad r = l - y$$

Thus

$$r^2 = x^2 + y^2 = l^2 - 2ly + y^2$$

i.e.

$$x^2 = l^2 - 2ly, \quad \text{where} \quad l = v_0^2/g.$$

This can also be written in the form

$$y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}.$$

This approach is by no means new, but provides an elementary approach to the problem if we do not wish to introduce the general concept of the envelope of a family of curves, obtained by the partial differentiation method.

Dennis C. Russell
University of London

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*Introduction to Mathematical Statistics,* second edition. By Paul G. Hoel, John Wiley & Sons, 440 Fourth Avenue, N. Y., May '53, 331 pp. $5.00.

Retaining the clarity of exposition that prompted its adoption by nearly 200 colleges and universities, the book now benefits from the many changes suggested by earlier users. An entirely new chapter on probability has been added, while the chapters on correlation and regression, the analysis of variance, and nonparametric methods have all been rewritten and expanded. The basic concepts of elementary statistical theory not only appear earlier in the book but with increased emphasis. As with the previous edition, only a background of elementary calculus is required.

Richard Cook

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*An Evaluation of Relativity Theory after a Half-Century.* By C. A. Muses, S. Weiser Inc., 117 Fourth Ave., N. Y.

Dr. Muses' study embraces several aspects of value both to technical and non-technical readers. It first of all provides an introduction to often little known historical aspects of relativity theory, throwing welcome light on the work of H. A. Lorentz, and its fundamental importance.

Secondly, it provides a clear-cut, understandable explanation of the theory in the light of its key concepts and their analysis.

Thirdly, modern contributions, such as the work of Hlavaty, are analyzed in terms of their usefulness and physical meaning, thus completing perhaps the most useful phase of a useful book; namely, its discussion of the directions in which the most advanced thinking on relativistic interpretation is tending.

Fourthly, certain original data are for the first time published, such as the relation of Cerenkov radiation to electronic aspects of relativity theory.

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